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ABSTRACT

This is one of a series that is a collection of translations from the extensive Soviet literature of the past 25 years on research in the psychology of mathematics instruction. It also includes works on methods of teaching mathematics directly influenced by the psychological research. Selected papers and books considered to be of value to the American mathematics educator have been translated from the Russian and appear in this series for the first time in English. The aim of this series is to acquaint mathematics educators and teachers with directions, ideas, and accomplishments in the psychology of mathematical instruction in the Soviet Union. The analysis of reasoning processes in the learning of concepts or the solving of problems is the theme common to the ten articles in this volume. These articles, except for the first one by Ushakova, were published between 1960 and 1967 and were part of the available literature during a revision of the Soviet school mathematics curriculum. The articles are interesting because of the topics they treat and because of the research styles they illustrate. (Author/MK)

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SOVIET STUDIES IN THE PSYCHOLOGY OF LEARNING AND TEACHING MATHEMATICS

VOLUME XIII

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Mary L. Charles
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SCHOOL MATHEMATICS STUDY GROUP
STANFORD UNIVERSITY
AND
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MATHEMATICAL LITERATURE
THE UNIVERSITY OF CHICAGO

SE 030 457

SOVIET STUDIES
IN THE
PSYCHOLOGY OF LEARNING
AND TEACHING MATHEMATICS

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VOLUME XIII

ANALYSES OF REASONING PROCESSES

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PREFACE

The series Soviet Studies in the *Psychology of Learning and Teaching Mathematics* is a collection of translations from the extensive Soviet literature of the past twenty-five years on research in the psychology of mathematical instruction. It also includes works on methods of teaching mathematics directly influenced by the psychological research. The series is the result of a joint effort by the School Mathematics Study Group at Stanford University, the Department of Mathematics Education at the University of Georgia, and the Survey of Recent East European Mathematical Literature at the University of Chicago. Selected papers and books considered to be of value to the American mathematics educator have been translated from the Russian and appear in this series for the first time in English.

Research achievements in psychology in the United States are outstanding indeed. Educational psychology, however, occupies only a small fraction of the field, and until recently little attention has been given to research in the psychology of learning and teaching particular school subjects.

The situation has been quite different in the Soviet Union. In view of the reigning social and political doctrines, several branches of psychology that are highly developed in the U.S. have scarcely been investigated in the Soviet Union. On the other hand, because of the Soviet emphasis on education and its function in the state, research in educational psychology has been given considerable moral and financial support. Consequently, it has attracted many creative and talented scholars whose contributions have been remarkable.*

Even prior to World War II, the Russians had made great strides in educational psychology. The creation in 1943 of the Academy of Pedagogical Sciences helped to intensify the research efforts and programs in this field. Since then the Academy has become the chief educational research and development center for the Soviet Union. One of the main aims of the Academy is to conduct research and to train research scholars

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*A study indicates that 37.5% of all materials in Soviet psychology published in one year was devoted to education and child psychology. See *Contemporary Soviet Psychology* by Josef Brozek (Chapter 7 of *Present-Day Russian Psychology*, Pergamon Press, 1966).

in general and specialized education, in educational psychology, and in methods of teaching various school subjects.

The Academy of Pedagogical Sciences of the USSR comprises ten research institutes in Moscow and Leningrad. Many of the studies reported in this series were conducted at the Academy's Institute of General and Polytechnical Education, Institute of Psychology, and Institute of Defectology, the last of which is concerned with the special psychology and educational techniques for handicapped children.

The Academy of Pedagogical Sciences has 31 members and 64 associate members, chosen from among distinguished Soviet scholars, scientists, and educators. Its permanent staff includes more than 650 research associates, who receive advice and cooperation from an additional 1,000 scholars and teachers. The research institutes of the Academy have available 100 "base" or laboratory schools and many other schools in which experiments are conducted. Developments in foreign countries are closely followed by the Bureau for the Study of Foreign Educational Experience and Information.

The Academy has its own publishing house, which issues hundreds of books each year and publishes the collections Izvestiya Akademii Pedagogicheskikh Nauk RSFSR [Proceedings of the Academy of Pedagogical Sciences of the RSFSR], the monthly Sovetskaya Pedagogika [Soviet Pedagogy], and the bimonthly Voprosy Psikhologii [Questions of Psychology]. Since 1963, the Academy has been issuing collection entitled Novye Issledovaniya v Pedagogicheskikh Naukakh [New Research in the Pedagogical Sciences] in order to disseminate information on current research.

A major difference between the Soviet and American conception of educational research is that Russian psychologists often use qualitative rather than quantitative methods of research in instructional psychology in accordance with the prevailing European tradition. American readers may thus find that some of the earlier Russian papers do not comply exactly to U.S. standards of design, analysis, and reporting. By using qualitative methods and by working with small groups, however, the Soviets have been able to penetrate into the child's thoughts and to analyze his mental processes. To this end they have also designed classroom tasks and settings for research and have emphasized long-term, genetic studies of learning.

Russian psychologists have concerned themselves with the dynamics of mental activity and with the aim of arriving at the principles of the learning process itself. They have investigated such areas as: the development of mental operations; the nature and development of thought; the formation of mathematical concepts and the related questions of generalization, abstraction, and concretization; the mental operations of analysis and synthesis; the development of spatial perception; the relation between memory and thought; the development of logical reasoning; the nature of mathematical skills; and the structure and special features of mathematical abilities.

In new approaches to educational research, some Russian psychologists have developed cybernetic and statistical models and techniques, and have made use of algorithms, mathematical logic and information sciences. Much attention has also been given to programmed instruction and to an examination of its psychological problems and its application for greater individualization in learning.

The interrelationship between instruction and child development is a source of sharp disagreement between the Geneva School of psychologists, led by Piaget, and the Soviet psychologists.* The Swiss psychologists ascribe limited significance to the role of instruction in the development of a child. According to them, instruction is subordinate to the specific stages in the development of the child's thinking--stages manifested at certain age levels and relatively independent of the conditions of instruction.

As representatives of the materialistic-evolutionist theory of the mind, Soviet psychologists ascribe a leading role to instruction. They assert that instruction broadens the potential of development, may accelerate it, and may exercise influence not only upon the sequence of the stages of development of the child's thought but even upon the very character of the stages. The Russians study development in the changing conditions of instruction, and by varying these conditions, they demonstrate how the nature of the child's development changes in the process. As a result, they are also investigating tests of giftedness and are using elaborate dynamic, rather than static, indices.

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*See The Problem of Instruction and Development at the 18th International Congress of Psychology by N. A. Menchinskaya and G. G. Saburova, Sovetskaya Pedagogika, 1967, No. 1. (English translation in Soviet Education, July 1967, Vol. 9, No. 9.)

Psychological research has had a considerable effect on the recent Soviet literature on methods of teaching mathematics. Experiments have shown the student's mathematical potential to be greater than had been previously assumed. Consequently, Russian psychologists have advocated the necessity of various changes in the content and methods of mathematical instruction and have participated in designing the new Soviet mathematics curriculum which has been introduced during the 1967-68 academic year.

The aim of this series is to acquaint mathematics educators and teachers with directions, ideas, and accomplishments in the psychology of mathematical instruction in the Soviet Union. This series should assist in opening up avenues of investigation to those who are interested in broadening the foundations of their profession, for it is generally recognized that experiment and research are indispensable for improving content and methods of school mathematics.

We hope that the volumes in this series will be used for study, discussion, and critical analysis in courses or seminars in teacher-training programs or in institutes for in-service teachers at various levels.

At present, materials have been prepared for fifteen volumes. Each book contains one or more articles under a general heading such as The Learning of Mathematical Concepts, The Structure of Mathematical Abilities and Problem Solving in Geometry. The introduction to each volume is intended to provide some background and guidance to its content.

Volumes I to VI were prepared jointly by the School Mathematics Study Group and the Survey of Recent East European Mathematical Literature, both conducted under grants from the National Science Foundation. When the activities of the School Mathematics Study Group ended in August, 1972, the Department of Mathematics Education at the University of Georgia undertook to assist in the editing of the remaining volumes. We express our appreciation to the Foundation and to the many people and organizations who contributed to the establishment and continuation of the series.

Jeremy Kilpatrick
Izaak Wirszup
Edward G. Begle
James W. Wilson

EDITORIAL NOTES

1. Bracketed numerals in the text refer to the numbered references at the end of each paper. Where there are two figures, e.g. [5:123], the second is a page reference. All references are to Russian editions, although titles have been translated and authors' names transliterated.

2. The transliteration scheme used is that of the Library of Congress, with diacritical marks omitted, except that Ю and Я are rendered as "yu" and "ya" instead of "iu" and "ia."

3. Numbered footnotes are those in the original paper, starred footnotes are used for editors' or translator's comments.

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INTRODUCTION

James W. Wilson and Jeremy Kilpatrick

The analysis of reasoning processes in the learning of concepts or in the solving of problems is the theme common to the ten articles in this volume. These articles, except for the first one by Ushakova, were published between 1960 and 1967 and were part of the available literature during the recent revision of the Soviet school mathematics curriculum. The articles are interesting because of the topics they treat and because of the research styles they illustrate. In particular, three of the articles each comprise a series of reports (published separately) showing a sequential attack on a particular research problem.

Ushakova conducted two investigations on the learning of visual concepts. In the first series of experiments her subjects were presented pairs of similar objects (leaves, pitchers, rectangles, lines) and then asked to reproduce them. The second series of experiments examined the effect of the presence of an auxiliary third object.

On first glance, Ushakova's paper seems to have very little to do with mathematics; much of the discussion concentrates on the richness of visual concepts of complex pairs of objects such as leaves or pitchers. It becomes clear, however, that although Ushakova approached the research questions like a psychologist, her interpretation of the results is directed toward classroom practice. The use of an auxiliary object to enhance a visual concept is a practical pedagogical tool. The regularities observed in the experiments can become expectations in the classroom, and the extent and exposure of visual comparisons can be tailored accordingly. The tasks that made use of lines and rectangles have clear relevance to mathematics learning.

The very brief paper by Davydov is a summary of his theoretical and experimental observations of children's formation of the concept of number. He explicitly restricts his attention to the process of developing number concepts during the time addition is being learned as a mental operation. The three consecutive stages in the process proceed from (1) adding quantities of things by counting the units (objective operation method), to (2) adding abstract quantities by counting the units (detailed verbal operation method), to (3) adding abstract quantities by counting the second addend onto the first taken as a whole (conceptual operation method proper). In these stages there are echoes of Piaget's conception of the genetic construction of the natural numbers [2], but Davydov highlights the role of counting and does not deal explicitly with class-inclusion or matching relationships. The training in the use of hand movements accompanied by slowed pronunciation seems to have had some effect in curtailing the counting process. It is not clear what effect such training might have on children's performance on Piagetian tasks related to the number concept.

Brushlinskii studied the problem solver's guidance of his thought processes during the course of problem solution. The mechanism described by Brushlinskii is the *generalized conception of a problem's solution*. This is a mental description by the subject of the general characteristics of the solution, omitting details. The general scheme is modified, or realized, in the course of solution--it guides the analysis and synthesis of the problem's conditions toward the solution. The subject begins by reading the problem and absorbing the problem's data. At the first stage of analysis of the problem's conditions the student forms a generalized conception. Further analysis yields concrete details of the conception from the problem's conditions, then modification of the conception on the basis of the concretezation, and hence progress toward a solution. One is struck by the similarity of Brushlinskii's construct of generalized conception and Polya's notion of a plan [3].

Shchedrovitskii and Yakobson studied the process of solving simple arithmetic problems in first grade and published a series of five reports. Logical analysis and observations of pupils were used to build a theoretical argument concerning problem-solving processes that rely heavily on counting and the use of objects. A thorough analysis is presented of how model building (with objects) can be used to solve simple arithmetic problems.

Kossov also studied first graders' processes of solving simple arithmetic problems but he had a different perspective than that of Shchedrovitskii and Yakobson. The series of seven reports by Kossov begins with an analysis of certain "non-switching" errors in arithmetic, where students continued to use an operation in a series of problems when in fact a new operation was indicated. Kossov identified a psychological regularity in that non-varying aspects of a stimulus series tend to become less strong. Hence in a series of addition problems the numbers change but the operation sign does not. Then when a subtraction problem is presented, the salience (or signalness as Kossov terms it) of the operation sign is so low that it is unnoticed by the pupil. This regularity was then examined in a variety of contexts and utilized to eliminate certain types of errors, facilitate the learning of sums less than 10, develop facility with solving simple arithmetic problems, and plan experimental instruction. The two final reports by Kossov dealt with a comparison of the effectiveness of alternative methods and the development of abstractions. The series of reports is especially illustrative of the development of a sequence of theoretical, empirical, and practical studies around a basic theme.

Mashbits investigated the solution of geometry problems by eighth graders. He was particularly interested in the formation of generalized operations as a method for problem solving. The students were classified as being in one of four stages, where each successive stage of forming operations was, relative to the preceding one, a higher level of generalization of the relationship between concepts. While the substance of this paper is similar to that of Brushlinskii, no link is acknowledged. Mashbits' theoretical discussion seems very much like an information-processing approach to psychology.

The article by Zavalishina and Pushkin draws from cybernetics and uses computer-generated problem solution sequences on a simple task to study sequences produced by human subjects. This is an example of the thorough analysis of a task structure preliminary to the study of students' solution attempts. The three forms of solution identified by Zavalishina and Pushkin correspond roughly to using trial and error, breaking the problem into parts, and solving the problem as a whole. The approach is similar to that used by Newell and Simon [1] in analyzing what they call a "problem behavior graph."

Artemov's investigations on the composition of pupils' geometry skills were contained in six reports published from 1963 to 1967. The series is an interesting progression from observational to highly theoretical investigations of various aspects of problem solving in geometry. The first three reports deal with relatively specific issues of instruction concerning auxiliary constructions, the concept of the plane, and using drawings, respectively. The second three reports deal with more general issues of instruction: the effect of drill on sets of exercises that are all of one type compared with drill on sets of exercises of varied types; the differentiation of instructional material from similar material introduced previously, and the effectiveness of a method of juxtaposition to overcome difficulties in making such a differentiation.

The second Brushlinskii article in this volume is a detailed logical analysis of the thought processes involved in problem solving. It is a critical analysis of other theoretical and pedagogical statements on problem solving. In particular, Brushlinskii argues for the inadequacy of heuristic rules such as those given by Polya [3] because of Polya's--and others'--lack of distinction, in Brushlinskii's view, between the requirements of a problem and what is being sought (the unknown).

The final article in the volume is a summarization by Fridman of a program of empirical research and the analyses of the logical-mathematical characteristics of arithmetic problems, leading to an extensive theoretical statement on the mechanisms for solving arithmetic problems. Object models, verbal models, and mathematically symbolic models of arithmetic problems are formulated and illustrated with examples.

These articles illustrate some of the range of Soviet interests in studies of reasoning processes. In particular, the series of related studies show how some programmatic research has been done that manages to bring together theoretical analyses, empirical data, and implications for instruction. The orientation toward instructional practice that distinguishes so much of Soviet educational and psychological research is particularly evident in the studies reported in this volume.

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2. Piaget, J. The Child's Conception of Number, London, Routledge and Kegan Paul, 1952.
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THE ROLE OF COMPARISON IN THE FORMATION OF CONCEPTS BY THIRD-GRADE PUPILS*

M. N. Ushakova

Introduction

Soviet psychology, according to the Leninist theory of reflection, defines concepts as images of objects or of processes (or of individual properties of them), which we do not perceive at a given moment. It is known from research that concepts,[1] which are reproduced images of reality, have a number of features to distinguish them from sensations and perceptions arising from the influence of reality over the sense organs.

In general paler than perception, _concepts can have a varying vividness_. In some persons, the visual concepts are the most vivid, in others the auditory concepts are most vivid, in still others the motor concepts are most vivid, and so on.

Visual concepts can have varying vividness, depending on the conditions of perception, the nature of the object, and its significance for the perceiver. We have had an opportunity to become convinced of this in asking schoolchildren to reproduce various objects from memory. The relative precision of the graphic reproduction and of the verbal description by the examinees gave us the right to draw conclusions about the significant differences in the vividness of the images. In some cases, the vividness of an image was close to the vividness of a perception; in other cases, it was distinguished by pallor and vagueness.

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[*] The research is a revised part of a doctoral dissertation, done under the supervision of I. M. Solov'ev. Published in _Proceedings [Izvestiya] of the Academy of Pedagogical Sciences of the RSFSR_, 1956, Vol. 76, 39-63. Translated by Joan W. Teller.

[1] In discussing concepts, we have in mind only concepts obtained as a result of the _visual_ perception of objects.

It is characteristic of concepts that individual parts and signs of the objects conceived are given with great vividness, others very dimly, and still others are altogether absent. The investigation we conducted provides proof of the appearance of gaps in concepts. Thus, a third grader's representation of a pitcher that was perceived previously was sometimes reproduced without a spout, without a bottom, or without a handle, and the leaf of a currant shrub was sometimes reproduced without lateral protuberances. All of these parts occurred, of course, in the originals. While it is generally accepted to regard all "gaps" in concepts as identical in their psychological nature, it is hard to agree with this. Our observations show there are essential differences between the "disappearance" of parts found on the periphery of an object and the "disappearance" of the object.

Let us cite examples of gaps in peripheral parts. Third graders sometimes reproduce the leaf of a currant shrub without lateral protuberances (Figure 1). They reproduce an earlier-perceived representation

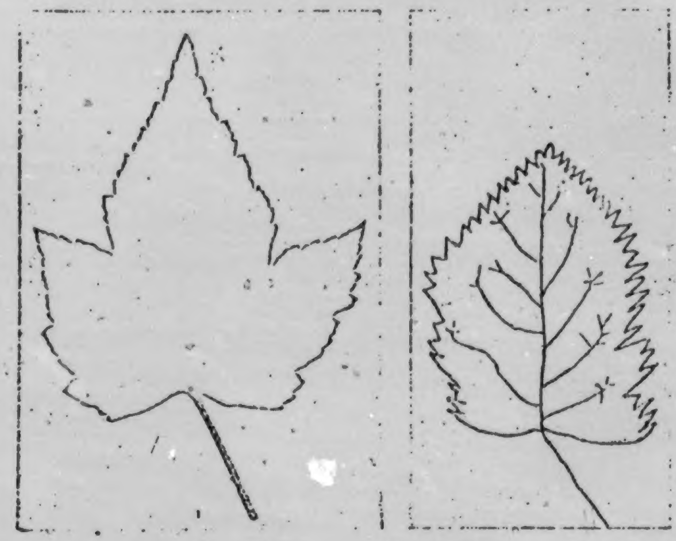

Figure 1

of a pitcher without the handle or the spout (Figure 2). The reproduction of a pitcher without the line separating the bottom from the body (Figure 3) is an example of the falling away of internal details.

The distinction between the cases consists in the fact that, in the disappearance of an internal detail, there are not gaps, in the true sense of the word, in the concepts. The object remains intact in the concepts, although more uniform (owing to the absence of detail). In the disappearance of border detail the object in the concept has a real gap. We note that in our experiments, the incompleteness of concepts increased with the complexity in structure of the objects. Reproductions of leaves showed more incompleteness than the reproduction of pitchers.[2]

An individual part, a detail of an object, sometimes is not fully absent in a concept, but turns out to be *smoothed over*. That is, the part may have lost its significance to some degree. On the other hand, a part may have been conceived in *accentuated* form, obtaining an unusual, exaggerated significance. According to our data, the smoothing over of parts and details of an object occurs quite often in concepts. In one of the investigations we conducted, third graders remembered and then reproduced in verbal description and graphic representation a birch leaf and a currant shrub leaf. In the reproductions obtained from the pupils the prominent teeth on the birch leaf were smoothed over in 83% of the cases, and the lateral protuberances on the currant shrub leaf were smoothed over in 50% of the cases. An exaggeration or accentuation of parts and details arose more rarely in the concepts and primarily with simpler objects. Thus the smoothing over of some parts and details can occur in concepts, while other traits are exaggerated or accentuated.

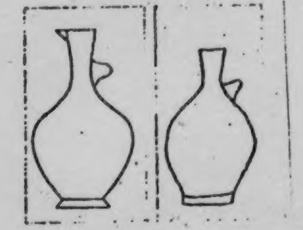

Figure 2

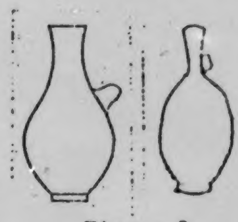

Figure 3

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[2] Our experiments confirm convincingly that children's concepts are notable for significantly less completeness than adults' concepts. This circumstance should be taken into consideration in work on the formation of schoolchildren's concepts.

The relative dimensions[3] of objects can be changed in concepts. Our experiments have shown that in the pupils' reproductions, objects, on the whole, are usually reproduced with some changes in proportion. The width was overestimated in some concepts, and the height in others. Thus, a currant shrub leaf was reproduced as broadened--the ratio of width to height was 98.1% in the reproduction, but 86% in the original. On the other hand, a birch leaf was reproduced as narrowed--the ratio of width to height of 51%, instead of the 54% in the original (Figures 4 and 5).

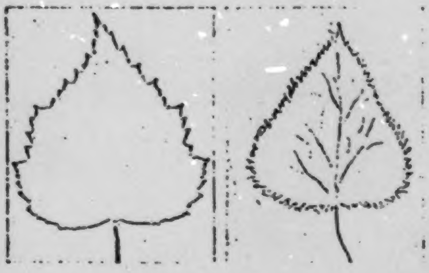

Figure 4

The concepts can differ from their respective objects on the basis of absolute size.[4] In the experiments we conducted with third graders, it came to light that in reproductions done immediately after presentation, the sizes of the objects studied, as a rule, were underestimated. Experiments showed that absolute dimensions were reproduced most accurately in the simplest objects (lines). With complication of the objects (rectangles, leaves), their absolute dimensions decrease considerably. The larger the size of the original, the more significantly its dimensions are decreased in reproduction. A tendency toward underestimation of objects in reproduction is so strong in children that it shows up not only in reproduction based on memory, but in the direct copying of objects as well.

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[3] We regarded the relationship of width to height, in each object conceived, as the relative dimensions.

[4] We regarded as absolute dimensions the length of the lines, as well as the areas of rectangles, pitchers, and leaves.

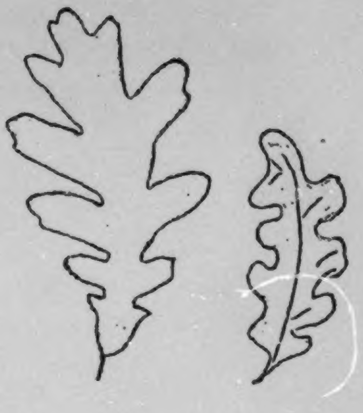

Figure 5

In our study of concepts of leaves, a general simplification of objects was manifested rather often in the concepts of third graders. This simplification was expressed both in simplifying the internal structure of a leaf (the venation) and in its exterior tracing. The frequency with which this simplification arose and its degree apparently depend on the complexity of the objects. Thus, in our experiments, leaves that were more complex in structure (the leaf of a currant shrub) were simplified more often and more intensively than simpler leaves (the birch leaf). It must be noted, however, that a perceived object can turn out to be more complex in the examinee's conception, in a number of cases. This complication occurs due to the appearance of details lacking in the originals. Evidently, the nature of the complications and additions is not accidental. It is conditioned by the child's past experience, the activity of his reproduction, the process by which a child strives to perfect a perceived object.

The study of concepts,[5] that is, images of objects absent at a given moment that influenced our sense organs earlier, assumes primarily the presence of a resemblance to the object. This resemblance, really, is the reflection of the object. Since they are reproduced images, however, concepts in the most complex conditions of the reproduction process often reflect the appropriate objects inadequately.

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[5] In examining the features of visual concepts, we did not touch on their generalized nature, supposing that a study of this question should be the task of a special investigation. We did not study the problem of general concepts either.

The facts cited correspond fully to the dialectic-materialistic understanding of concepts as secondary images of the reflection of reality. From this point of view it becomes entirely clear that only in extremely rare cases can concepts have vividness in perceptions, completeness in them, stability, wholeness; that in the overwhelming majority of cases, the concepts (especially the concepts of younger pupils) are paler than the perceptions, unstable, and incomplete. As testimony to this there is the important and laborious work of the best teachers in creating in pupils vivid, complete, and stable concepts of the objects and phenomena that are studied.

It should be said that all the noted traits of concepts are expressed particularly sharply in children of young school age. Our experiments showed that in the concepts of young pupils, smoothing over and accentuating the features of objects, underestimating the size of the originals in reproductions, and the diffusion of the internal structure of objects, are manifested more sharply than in the reproductions of adults. The augmentation and complication of objects are also found more often and more significantly in the reproductions by pupils. Moreover, the great general variability with time of the pupils' concepts should be noted. Therefore, it is especially important that teachers achieve not only an adequacy of younger pupils' concepts of the objects perceived, but durability of these concepts as well.

Psychological investigations of the reproductions of objects over various intervals of time after perception have shown that they do not repeat one another; thus the regularity of the changes occurring in them has been revealed. A detailed analysis has shown that the reasons for these changes, on the one hand, lie in the deformation of the images themselves under the operation of time and, on the other hand, it was clarified that changes in images can arise in the process of reproduction under the influence of external conditions--for example, changes in the goal or task of reality.

Soviet psychologists (I. M. Solov'ev and others) have studied in the greatest detail two interrelated directions of change of concepts: the accentuation of features of the new object in a concept (the exaggeration of its differences from old, more familiar objects), and the likening of a new object (smoothing over features, an increase in likeness to a well-known, old concept). The experiments showed that likening is a more extensive and profound change in concepts, which supersedes the initial accentuation.

In teaching in school, teachers use different methods for creating in the pupils sufficiently complete, accurate concepts of the objects studied. The most widespread of these methods are the demonstration of an object in a lesson, accompanied by the teacher's explanations; the representation of an object on the blackboard by the teacher; sketching it in notebooks by the pupils; and the repeated perception of an object by the pupils. While recognizing the importance and advisability of using these methods in teaching, it should be noted that some of these methods (drawing) take up too much time, which does not allow them to be used often; others (the repeated demonstration of an object) are not always effective. Nearly the most effective and most convenient, but, unfortunately, an insufficiently used means which makes a concept precise is the comparison of objects in perception.

It would be a profound mistake to suppose that comparison is brought about equally, regardless of what we compare and for what purpose we make the comparison. On the contrary, it is established, for example, that comparison of perceived objects has features that distinguish it from the mental comparison of conceived objects. According to our observations, the nature of comparison changes depending on the purpose. But the purposes of a comparison can be quite diverse. In some cases, comparisons are used to define more precisely concepts that are already familiar or known; in others, comparison is used for elaborating new concepts. Comparison can also be used for memorization, reproduction, generalization, abstraction, and the like. Comparison, finally, is used for the classification and systematization of ideas and concepts. The process of comparison appears as an essential means of forming a system of ideas, elementary concepts in general, or a system of knowledge in schoolchildren.

In a study of the question it is impossible to overlook the major contribution to the methodology and psychology of comparison that Ushinskii has made. Without entering into a critical examination of Ushinskii's studies of comparison, we should note that the great Russian educator and psychologist showed the significance of comparison in the process of cognition of reality, as well as the role of comparison in teaching children the foundations of the sciences. He wrote:

> *It is through comparison that we get to know everything in the world* [our italics], and if some new object is presented to us, which you cannot equate with anything or differentiate from anything (if such an object were possible), we could form no thought about this object, nor could we say one word about it. If you want some object of external nature to be quite clear, then differentiate it from the objects that are most like it and find similarity in it with the objects that are most remote from it: only then will you find out all the essential features of the object, and this means understanding the object [7:436].

It must be noted that, unlike the majority of his contemporaries, Ushinskii's views contained elements of a dialectic approach to a settlement of the problem of comparison. Thus, Ushinskii wrote: "...Both the feeling for similarity and the feeling for difference are only two sides of the same process--the process of comparison [6:472]."

In valuing comparison highly for didactics, Ushinskii showed the variety and wealth of opportunities for using comparison in instruction. Even though the most progressive teachers use comparison in their lessons, however, there are no united demands for the use of the process of comparison. Even for elementary school a methodology of the comparison process has not been worked out; there is no selection of objects for comparison according to topics.

The authors of the best contemporary methodologies, in stressing the importance and necessity of comparison in instruction, unfortunately do not give methodologically elaborated indications for the organization and conduct of comparison--in particular, advice on what to compare or in what order. Nevertheless, these features are quite essential, since the result of comparison depends on them.

The importance of the problem of comparing objects[6] prompted us to conduct special experimental investigations of the comparison process in the third grade of school. We expected to show that comparison will promote the exposure of the distinctiveness of the objects and a more precise reproduction of these objects on the basis of memory.

Investigation I. The Influence of Comparison of Two Objects on Accuracy in Reproducing Them

The task of the present investigation was to establish: What (concrete) influence is exerted by comparison of perceived objects on the quality of their subsequent reproduction? How does the quality of reproduced images depend on the comparison of objects that are similar in a varying degree? How is it possible to improve, to perfect comparison, so that it promotes the rise of concepts that reflect the objects more accurately?

Method

The examinees--third grade pupils--were given dried leaves from the herbarium, attached to cards (8.7 x 13 cm) and covered with cellophane for preservation; outline representations of two pitchers, two rectangles, and two lines, drawn in India ink on cards of white paper (7.2 x 10.8 cm).[7]

The experimental material was presented in pairs composed in the following way:

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[6] Beginning with 1940, in Soviet psychology the question of the process of comparison in pupils was investigated especially by I. M. Solov'ev, Zh. I. Shif, M. M. Nudel'man, A. A. Smirnov, N. P. Fester, M. V. Zvereva, V. E. Syrkina, A. I. Lipkina, R. I. Ivanov, T. D. Kirillova, and others.

[7] The objects selected were taken from elementary school practice. Thus, the leaves were chosen in connection with the fact that, beginning in grade 1, the children should become familiar with various aspects of trees; the geometric figures were chosen because an elementary study of geometry is begun in grade 3--the children become acquainted with the simplest geometric figures (rectangles, triangles, lines) and start the study of areas.

Pair I -- a birch leaf and a currant shrub leaf (Figure 6);
Pair II -- two oak leaves, differing in dimensions and shape (Figure 7);
Pair III -- two pitchers of different shape (Figure 8);
Pair IV -- two rectangles, differing in width (Figure 9);
Pair V -- two straight lines of different length.

The experiment proceeded as follows. The pair of objects was presented to each examinee. Each object was presented for 10 seconds. Three seconds after the first object of a pair was taken away, the second was presented. The reproduction followed immediately after the exposition of the second object in each pair. The time for reproducing was not limited. The children were told beforehand that they could correct and improve their drawings for as long a time as they needed. When a reproduction of the first object in a pair was finished, the experimenter took away the completed drawing, and the examinee reproduced the second object.

Before each experiment, the examinees were asked to examine carefully the objects that were shown, to compare them with one another, to establish similarity and difference, and to try to remember them as well as possible in order to reproduce the objects in the same sequence as they were presented, as accurately as possible. After the reproduction of each pair of objects, a discussion was held with the examinee, in which the sort of objects he saw was ascertained. Were they similar to one another or dissimilar? How were they similar? How dissimilar? The experiments were conducted with 30 pupils in third grades in Moscow. Control experiments were conducted with a group of adults.

Results of the experiments

Does the comparison of objects by pupils in third grade of the mass school promote an improvement in the quality[8] of the appropriate concepts?

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[8] By the quality of concepts we mean the degree of accuracy of the reflection of an object.

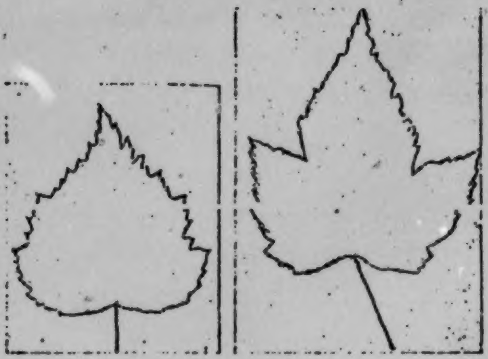

Figure 6

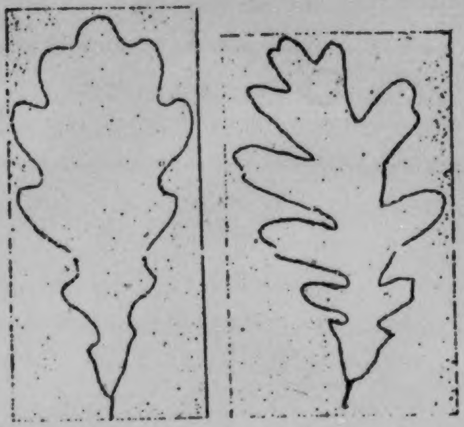

Figure 7

To answer this question, we must compare the results of reproducing the same objects when perceived in isolation without comparison and when compared during perception.

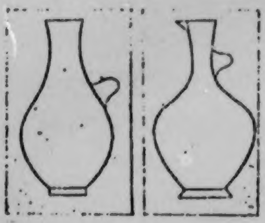

Figure 8

Figure 9

The experiments we conducted permit us to assert that the comparison of objects during perception exerts a positive influence on the reproduction of the objects compared, by pupils in third grade. The reproductions obtained with comparison during perception contain features and parts of objects omitted by pupils reproducing objects perceived in isolation. If the lateral parts of a currant shrub leaf were missing in the reproduction without comparison, then after comparing the leaves of a birch and a currant shrub, in the reproduction of the currant shrub leaf there were wide lateral protuberances with sharp tips, and typical of both leaves was a cut in the graft. The reproductions of oak leaves were also more complete: in the second oak leaf, (in order of presentation), a new, large, lateral tooth (4), a graft, was added after comparison (see Figure 10, a and b).[9] In the reproduction of representations of pitchers after comparison a spout appeared on the second pitcher in the pair, and at the bottom of the second pitcher, a widening of the lower part was added (see Figure 11, a and b).

Smoothing over of parts and details in reproductions after comparison diminished, and features that were smoothed over earlier in objects began to be reproduced more expressively. Experiments showed that in this case in the reproductions of birch and currant shrub leaves, the serrations in both leaves began to be reproduced more expressively, grooves appeared or became deeper in places where grafts were attached, an incline of the tip to the left appeared in the birch leaf. In the reproduction of pitchers after comparison, the spout of the second pitcher became pointed, and the lower part of the bottom of this pitcher also became pointed.

Comparison appreciably improved the reproduction of the general outlines of the objects. The examinees' reproductions show that the proportions of the objects after comparison were reproduced more accurately in 60 percent of the cases. Thus, after comparison the examinees reproduced the relationship of width to height more accurately in five (out of eight) objects. (Currant shrub leaf B, oak leaf A, pitchers A and B, rectangle A.)

The reproductions we obtained from the examinees also allow us to establish that comparison exerted a favorable influence on the reproduction of absolute dimensions in five out of ten objects (Table 1).

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[9] In Figure 10a an oak leaf is represented after an isolated reproduction. Figure 10b is a reproduction of the same leaf after comparison with another oak leaf. Both drawings were done by a single examinee.

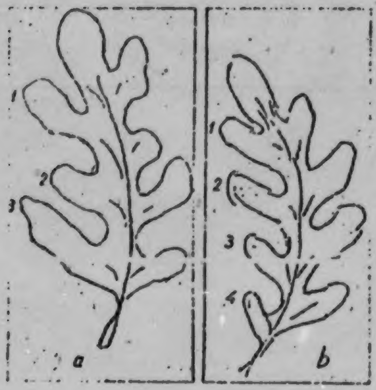

Figure 10

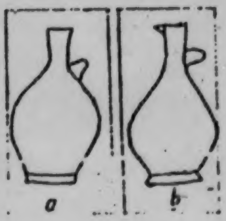

Figure 11

TABLE 1

Percent of Absolute Dimensions
in Reproductions

Number of Examinees	Type of Reproduction	Currant Leaf 5226 mm^2	Oak Leaf A 5994 mm^2	Pitcher A 1326 mm^2	Pitcher B 1479 mm^2	Line B 43 mm
20	Reproduction without Comparison	50	54	65	81	94
20	Reproduction after Comparison in a Pair	58	59	80	86	96

Table 1 shows, however, that the absolute dimensions of the objects in the drawings continue to remain significantly smaller than in the originals.

The experiments showed that comparison assists the appearance of the parts of objects that have previously been lacking in pupils' reproductions, leading to a more expressive representation of the parts and details of objects (a decrease in smoothing over, an increase in breaking down into details). In some cases comparison also turned out to be useful for a more accurate reproduction of both absolute and relative dimensions of the objects compared. In general it can be said that the reproductions of objects after comparison approached the models in some, and often in all, respects.

This conclusion prompted us to ask the question: Does the result of comparison depend on the nature of the objects compared? Would it be identical in the comparison of any objects? Would it change, for example, according to whether the objects compared are similar or dissimilar?

We considered it desirable to obtain numerous data on this question, and we therefore used simpler objects, a change in the reproductions of which lends itself to a sufficiently precise measurement. The reproductions of dissimilar pairs (pitchers--lines) and similar ones (two pitchers--two lines) were studied.

TABLE 2

Absolute Dimensions of Reproductions of Pitchers Under Similar and Dissimilar Comparisons

Number of Examinees	Comparison in a Pair	Pitcher A[a]		Pitcher B[b]	
		Mean Percent of Width to Height	Deviation from Original	Mean Percent of Width to Height	Deviation from Original
10	Pitcher—line	55.7	+4.7	66.4	+9.4
10	Two pitchers	55.2	+1.2	59.8	+2.8

[a] Pitcher A: Percent of width to height in the original—51.

[b] Pitcher B: Percent of width to height in the original--57.

TABLE 3

Absolute Dimensions of Reproductions of Lines Under Similar and Dissimilar Comparisons

Number of Examinees	Comparison in a Pair	Line A[a] (Arithmetic mean)	Deviation from the Original	Line B (Arithmetic mean)	Deviation from the original
10	Pitcher--line	29.8	-2.2	41.0	-2.0
10	Two lines	30.3	-1.5	41.4	-1.6

[a] Length of line A in the original--32 mm, of line B—43 mm.

The data in Tables 2 and 3 show that the reproductions of dissimilar objects deviate significantly from the originals. Similar objects were reproduced much more accurately. Consequently, the results of comparison depend on the objective relationships of the objects compared; in particular, the results depend on the degree of similarity of the objects to be compared.

It was established in Pavlov's experiments that an inhibiting process lies at the basis of differentiation. Here--Pavlov points out--"...the higher the degree of differentiation inhibition, that is, the closer the differential agents are to each other, the more significant is the successive inhibition, all other conditions being equal [4:139]."

The positive role--established in our experiments--of comparison in the process of exposing the features of objects finds confirmation in Pavlov's research. He writes:

> How does the specialization of a conditioned stimulus, the differentiation of external agents, proceed? At first we thought that two devices take place here. One was only the multiple repetition of a definite agent as a conditioned stimulus.... The other was the intermittent contrasting of this definite, constantly reinforcible, conditioned stimulus with an agent close to it.... At present we are inclined to admit the validity only of the latter device [4:130].

The experimental investigation that has been cited permits us to establish that comparison of objects in the process of perception exerts a positive influence on the reproduction of objects by third graders and that the influence of comparison depends on the objective conditions in which the objects to be compared were placed. The experiments showed that the result of comparison is different in the reproduction of pairs of more similar and of less similar objects.

The peculiar traits inherent in given objects are exposed in the best form in the comparison of the most similar of them. Thus, the peculiarity of the general shape and venation of a leaf was best reflected in the reproductions of similar leaves from an oak; the peculiar serration of the edges was especially expressively reproduced with the leaves of a birch and of a currant shrub, which are most like each other in this respect.

The peculiarity of proportions was more accurately reflected in drawings of objects more similar in that respect (pitchers); it was revealed more weakly in objects whose proportions differed significantly (the leaves of a birch and of a currant shrub).

It turned out that the quality of the reproduction of an object depends on the place the object occupies in the pair to be compared. Our investigations showed that, in successive comparison, if the interval of time between presentations of objects is small, the first object in a pair is reproduced best. It should be borne in mind, however, that with an increase in the time-interval between presentations of objects, the quality of reproduction of the second member of a pair begins to be heightened.

In the analysis of the pupils' reproductions, a dependence of the result of comparison on the complexity of the objects compared was clearly revealed. This dependence became apparent in the fact that the favorable influence of comparison in the sense of an increase in completeness and of the elimination of smoothing over in an object in a representation, affected chiefly the reproduction of complex objects. Of course, there is a question as to the degrees of complexity that the objects we studied possessed.

All of the proofs cited above for the positive influence of comparison on the quality of concepts should not hide the important circumstances that in many cases of the comparison of objects there were not enough examples for making concepts more precise.

The comparison of objects in a pair in some cases did not fully help to overcome incompleteness, smoothing over, and the changeability of proportions and of absolute measurements of objects in the concepts of young pupils. And what is more, it sometimes even resulted in a deterioration in the quality of the images of objects because a likening of similar concepts arose. This caused features of the objects to be smoothed over and the similarity between the objects in a pair increased significantly and illegally. In some cases the particular and distinctive traits of an object were sharply exaggerated, accentuated as a result of comparison, and the proper representation of objects was also thus impaired in the concepts.

Investigation II. The Influence of Comparison of Two Objects with a Third on the Formation of Concepts

A comparison of two objects is not always enough for the creation of complete, accurate concepts of these objects. Some essential features can remain unrevealed. Observations show that in these difficult cases teachers ask the pupils to compare the objects in a pair with a third object.[10] Practice shows that the skillful use of this technique, which promotes the exposure of hitherto unnoticed features of the objects being compared, at the same time promotes an understanding of the actual relationships of the objects compared and thus assists the mastery of a system of knowledge.

Observations of lessons have shown that teachers seldom use this technique. Only educators who work most thoughtfully use it systematically in teaching botany and zoology (e.g., the teacher Gavrilov in the Kalinin Suvorov Military School). However, even the best teachers use this technique with an insufficient realization of its significance.

We believe that a psychological study of this technique is quite timely. It is entirely clear that in the process of this comparison the pupils experience new ties and relationships between the objects to be compared, and they interpret them more deeply.

Let us note that in Soviet psychology, Zh. I. Shif [5] was the first to become interested in this fact, in 1941, and six years later, N. P. Ferster [1]. Shif found that the use, for comparison, of a pair of auxiliary objects facilitates the singling out of features in the process of comparison and promotes the exposure of the relationships between objects that could not be exposed in comparing them in a pair. Ferster confirmed the basic conclusions that Shif obtained, but the investigations they conducted did not affect the question of perfecting concepts with the aid of this technique.

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[10] Thus, in order for seventh graders to understand the similarity between the brains of reptiles and of fish, N. P. Panchenko, a teacher in School No. 114, asked them to compare preparations of reptile- and fish-brains with the brain of a bird, characteristic of which is the presence of a cerebellum.

The following questions confronted us:

1. Does the introduction of an auxiliary, "third" object influence the conception of objects in a pair?

2. Does comparison with an auxiliary object always yield a positive result?

3. Does the influence of the "third" object depend on what object is used as auxiliary--in particular, how does the influence of the auxiliary object depend on the extent of its similarity with objects in the pair to be compared?

4. Does introducing an auxiliary object promote the exposure only of similarity, or can it influence the isolation of differences?

5. Can the indicated technique serve as a means of perfecting concepts?

Our task was to investigate and to prove the many-sided influence of auxiliary objects on the comparison of objects in a pair, and to show the dependence of that influence on the relationships entered into by the third object with the objects in the pair.

The educational significance of the study of this question is completely obvious. Even a partial solution to it will put into the teacher's hands a didactic means for creating accurate concepts of objects.

Method

A month after the first reproduction, described in Investigation I, the same third graders were asked to copy and to remember a third object, similar in varying degree to the objects of each pair to be compared, and only afterwards to compare and reproduce from memory the pairs of leaves, pitchers, rectangles, and lines (Figures 12, 13, 14). The order of presentation of the objects in a pair, the time for exposition of each object, and the interval between them remained unchanged in all cases. In all the series, reproducing was done immediately after presenting the pair.

The auxiliary, "third" objects were objects of the same type, specially selected for each pair on the basis of shape and dimensions.

Assuming that the influence of the auxiliary objects would be determined by the degree of their difference from the objects to be compared, there were specially selected as auxiliary, "third" objects for the first pair--a linden leaf, more similar to the first object in the pair (a birch leaf), for the second pair--an ash leaf, more similar to the second, the oak leaf. For each pair of simpler objects (representations of pitchers, rectangles, lines), some auxiliary objects were selected, differing, in varying degrees, in width and length from the basic objects in the pair.

Thus, 17 series of comparisons with auxiliary objects were conducted, some of which were smaller than the first object in the pair, and others of which were larger than the second object in the pair.

By having the examinees copy the third object before comparing a pair, we expected to elicit its influence and to determine the nature of that influence.

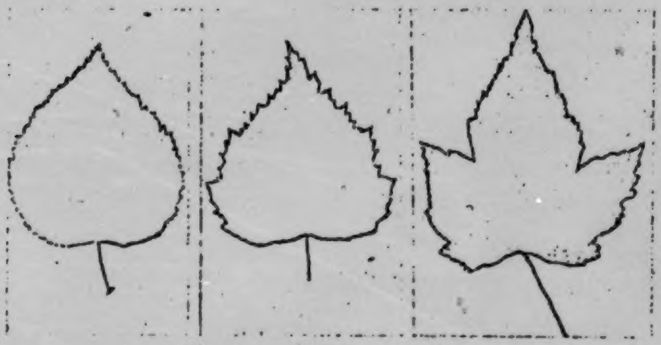

Figure 12

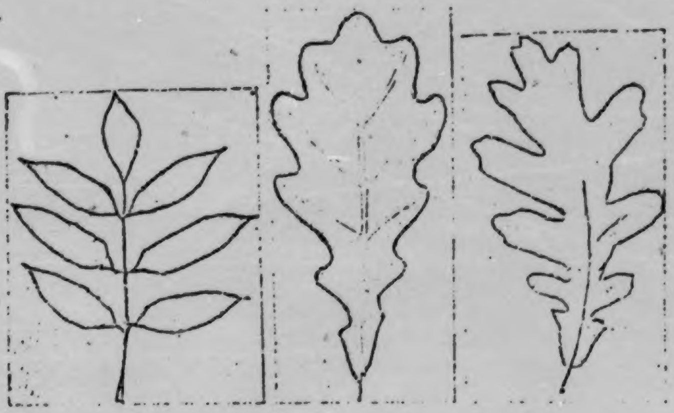

Figure 13

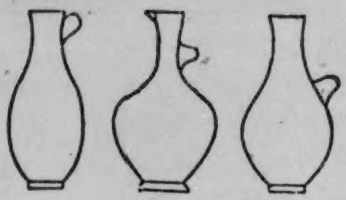

Figure 14

Results of the experiments.

Introducing an auxiliary object into the comparison of two objects exerted an influence on the reproduction of all of the objects. This influence was more significant where the third was more similar to the elements in the pair.

The reproductions we obtained from the examinees show that copying a third leaf (a linden or an ash leaf) before comparing a pair promoted a clarification of the concepts of the leaves in both pairs. The children's drawings show that this kind of comparison with a "third" object significantly helps to eliminate defects in the reproductions that could not be eliminated when a pair of leaves were compared. Thus, after comparison with a "third," the representations of the leaves in both pairs became more complete. In these reproductions parts of the currant shrub leaf (lateral protuberances) and of the oak leaves (teeth, a broad, rounded tip) appeared that had been lacking in some cases during comparison in a pair.

The smoothing over of features and the accentuation of them, which occurred during comparison in a pair, disappeared in the representation of birch and oak leaves. The general shape, the serration, and the venation of the leaves began to be reproduced more accurately after copying a third leaf and the distinctiveness of each leaf in a pair was significantly better expressed in reproduction. Thus, in the examinees' reproductions, the birch leaf began to be represented more accurately in its general shape; it became widened in the lower part and acquired its characteristic triangular shape, the shape of the original, and its pointed tip stood out clearly (Figure 15a_1). The large lateral protuberances in the currant shrub leaf also began to stand out more distinctly, and the leaf was broken up more (Figure 15b_1). The serrate edges of the leaves in both pairs began to be reproduced more accurately after a third leaf was copied. On the birch leaf the characteristic large teeth appeared on each side. In the reproductions of oak leaves, smoothing over the outline gave way to a considerable breaking down into parts—especially, the serrations of the second leaf in the pair became deeper, sharply expressed, while the outline of the first leaf, on the other hand, became somewhat rounded, as happened in the models.

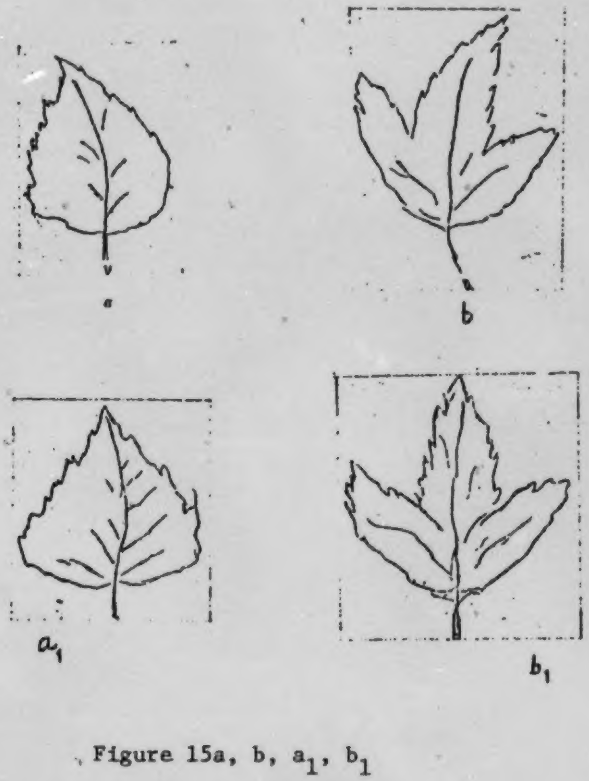

Figure 15a, b, a_1, b_1

Introducing a third leaf in a comparison also exerted a positive influence on the reproductions of the leaves to be compared in the respect that superfluous additions, which arose in reproductions when two leaves were compared, disappeared afterwards. Thus, in the currant shrub and oak leaves (the second leaves in each pair) the superfluous, large lateral protuberances and teeth disappeared. The absolute dimensions of the leaves in both pairs were reproduced more accurately (see Table 4).

TABLE 4

Percent of Absolute Dimensions of Reproductions

Number of Examinees	Size of the originals in mm^2	Birch Leaf 3596 mm^2	Currant Leaf 5226 mm^2	Oak Leaf 5994 mm^2	Oak Leaf B 5610 mm^2
20	Reproductions after comparison in a pair	49	58	59	57
20	Reproductions after comparison with an auxiliary object	65	73	68	72

Note. The areas of the circumscribed rectangles in the originals and in the reproductions were compared to obtain these percents.

In the analysis of the reproductions it was revealed that copying a third leaf changed the nature of the mutual influence of the leaves to be compared in each pair. The influence of the third object, which was spread over both objects in a pair, affected differently each of the leaves to be compared, depending on the degree of objective similarity of the third to the individual leaves of the pair. Thus, copying a linden leaf influenced the reproduction of both leaves to be compared, but it had the greatest positive influence on the birch leaf, which was more like it in its general shape.

In conformity with this, we obtained, in our pupil-examinees' reproductions, a more accurate representation of the general shape of the birch leaf in 86 percent of the reproductions and of the currant

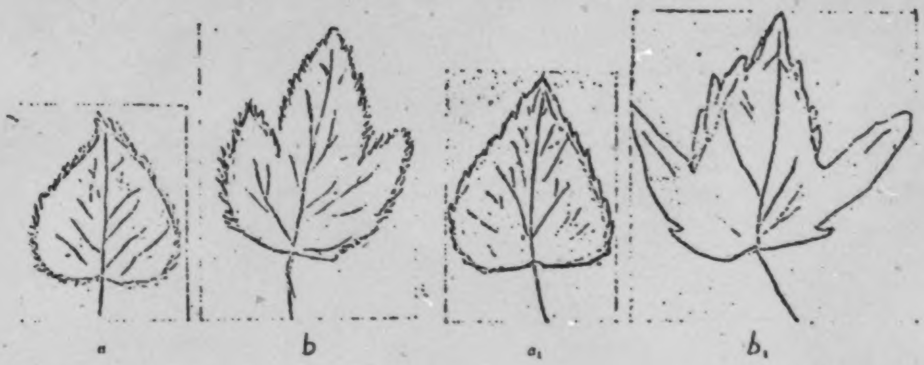

Figure 16 a,b, a_1, b_1

shrub leaf in only 60 percent of the reproductions. On the basis of the serrations, the birch leaf began to be reproduced more accurately in 80 percent of the reproductions and the currant shrub leaf in only 46 percent.

Our investigation showed that in some cases an auxiliary, third leaf in a comparison aided the judgment of similarity between a pair of leaves, whereas in other cases, introducing it evoked an accentuation of the difference between them. In an analysis of the results it was established that the influence of the auxiliary object was determined by the degree of its similarity with the objects in the pair. Introducing an auxiliary leaf (of a linden) that was closer to the first leaf in the first pair (the birch leaf) evoked an emphasis of the difference between the birch and currant shrub leaves in reproduction; introducing an auxiliary leaf (of an ash) significantly different from both leaves in the second pair promoted an increase in similarity between the oak leaves in the examinees' reproductions. Thus, it can often be noticed, in the records of the statements made, that in comparing a birch leaf and a currant shrub leaf, the examinees exposed first and foremost the similarity between these leaves, and only after an indication of the similar features did they turn to exposing the features of difference. It is curious that after the introduction of a third object--a linden leaf--the examinees first discovered the difference in the leaves to be compared, and only afterwards, and not in so much detail, did they notice similarity. Thus, the examinee Natasha L. said that "after comparing the leaves of a currant shrub and a birch with a linden leaf, I noticed that they differed." And she added that "the linden leaf helped me to draw and to remember the birch leaf."

When an examinee was given similar oak leaves for comparison in a pair, the introduction of a third--an ash leaf--that differed significantly from both elements in the pair, promoted an exposure and even an exaggeration of the similarity between the leaves in the pair. In the result the ash leaf had less influence on the reproduction of the features of both oak leaves, with its influence being negative in a certain respect.

The experiments showed that here the influence of an auxiliary object on the individual objects in a pair was different. In general shape and serration of the edges, the ash leaf was closer to the second oak leaf, although it differed more from it than did the linden leaf from the birch leaf. The experiments showed that after copying the ash leaf, the representation of the general shape and serration of the second oak leaf was improved (the representation of the general shape improved in 86 percent of the cases, and of the serration in 80 percent), while the representation of the corresponding features of the first oak leaf improved, but to a significantly lesser extent (the transfer of the general shape of the first leaf improved in only 40 percent of the cases, and of the serration in 33 percent of the cases).

After copying the ash leaf, the examinees, naturally, compared it first with the first oak leaf, but on account of its considerable difference, the "third" object did not exert a strong positive influence here. When the examinees were given the second oak leaf, they established its similarity with the ash leaf, which promoted the exposure of peculiarities and resulted in an improvement in the reproductions of the second oak leaf.

Thus, the experiments with leaves give us the right to assert that introducing auxiliary objects exerts an influence on the reproduction of the leaves in each pair to be compared. The auxiliary object significantly furthered the accuracy in reproduction of leaves to be compared in a pair, with respect both to shape and to dimensions. The influence of the auxiliary object depended on the degree of its similarity with the objects in a pair.

Our experiments showed that not only can introducing an auxiliary object promote the exposure of similarity, as Shif and Ferster indicated, but in some cases such as introduction can promote the exposure of differences between the objects in a pair. It was also established that in comparison with a third object, more subtle and varied relationships of similarity and difference are disclosed between the objects to be compared.

However, the results obtained were not able to satisfy us, since, in using such complex objects as leaves, it turned out to be very hard to reveal basic laws governing the influence of auxiliary objects. To check, specify, and find a numerical expression of the laws that were found, we conducted experiments in the comparison of simpler objects—namely, lines and outline representations of rectangles and pitchers.

According to the method set forth above, we conducted seven series of experiments in the comparison of lines, which differed only in the dimensions of the auxiliary line. In some experiments auxiliary lines were introduced that were smaller than the lines in the pair, differing from the first line by 12, 30 or 50 percent. In other series of experiments, auxiliary lines were introduced that exceeded the lines in the pair in length, differing from the second line in the pair by 12, 30, 40, or 50 percent. Analogous experiments were conducted with rectangles and pitchers.

We were interested in the extent to which the influence of an auxiliary, third line affects the relationship between the basic lines that were to be reproduced in a pair, and in their absolute dimensions as well. Table 5 was composed to answer the first of these questions.

TABLE 5

Variations in Reproductions of a Pair of Lines When Compared a Shorter Auxiliary Lines

Series Reproduction (30 examinees)	Difference between Auxiliary line and line A in percent of A	Deviation of ratio in reproductions (ratio of B to A)
After comparison of the pair	---	+1.3
After comparison with an auxiliary line	12 percent	-1.5
After comparison with an auxiliary line	30 percent	+1.9
After comparison with an auxiliary line	50 percent	-0.5

The figures in the third column show the deviation in the ratio between the lines in the reproductions. The ratio (percent, line B of line A) of the lines that served as models was 134.4. An increase of the ratio when an auxiliary line was introduced is testimony that the difference between the lines is increased; a decrease in the ratio reveals a smoothing over of difference, a likening of the lines in a pair.

It was revealed in the experiments that the relationships between the lines, rectangles, and pitchers to be compared, and the absolute and relative dimensions[11] of these objects changed regularly after auxiliary objects were introduced in the examinees' concepts--depending on the degree of similarity between the "third" object and the objects in a pair.

The data in Table 5 show that the introduction of an auxiliary line that is closer in dimensions to line A (the first object in the pair) evoked a *smoothing over* of the difference between the lines of the pair (deviation of -1.5), a trimming of their lengths. With an increase in the difference (from 12 percent to 30 percent) between the auxiliary line and line A, an *accentuation* of the difference between the lengths of the lines to be compared was observed (deviation of +1.9). When the difference between the auxiliary line and line A of the pair was increased even more (50 percent), a *smoothing over* of the difference again ensued, a trimming of both lines in length (deviation of -0.5)

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[11] We regarded absolute dimensions to be the length of the lines, and the areas of rectangles, pitchers, leaves; relative dimensions were the ratio of width to height in each rectangle, pitcher, or leaf.

TABLE 6

Variation in Reproductions of a Pair of Lines When Compared with a Longer Auxiliary Line

Series Reproductions (30 examinees)	Difference between auxiliary line and line B in percent of B	Deviation of ratio in reproductions (ratio of B to A)
After comparison in a pair	---	+1.3
After comparison with an auxiliary line	12%	+4.5
After comparison with an auxiliary line	30%	-0.7
After comparison with an auxiliary line	40%	-3.7
After comparison with an auxiliary line	50%	-5.7

When lines exceeding the second line of a pair in length were introduced into an experiment (see Table 6), the auxiliary line that was closest in dimensions to line B (the second object in the pair) elicited an *accentuation* of the difference between the dimensions of the lines in the pair (deviation of +4.5; see Table 6). With an increase in the difference between the auxiliary line and line B, accentuation of the difference gave way to a *smoothing over* of it, and the lines in the pair began to be trimmed in length (deviation of -0.7). A still greater difference between the auxiliary line and line B caused the distinction between the sizes of the lines to be compared to be smoothed over even more (deviations of -3.7, -5.7)--they were trimmed even more.

It should be noted that the general rule for the influence of an auxiliary object noted in the comparison of lines was confirmed during the comparison of the representations of rectangles and pitchers. Depending on the degree of similarity between the auxiliary object and the corresponding objects in a pair, in some cases of reproduction an increase was observed in the difference in size, in general shape or in features of the outline of the rectangles and pitchers to be compared. In other cases, however, the rectangles and pitchers in a pair in the examinees' drawings became more alike in dimension and shape than in the models.

It was established in the investigation that introducing an auxiliary object promotes the comparison of a pair under definite conditions. If the difference, for example, between the third line and the first or second of the lines to be compared in a pair does not exceed a definite size the comparison is improved. This difference between the auxiliary object and the pair to be compared, within the limits of which this object exerts a positive influence on the comparison process, we call the zone of optimal difference. The experiments showed that the more complex the objects to be compared, the closer to the first of them, within certain limits, was the zone of optimal difference of the auxiliary object that makes the relationship between the objects in the pair more precise.

An analysis of the results showed that changes in the objects in a pair in the examinees' concepts after the introduction of auxiliary objects were evoked by a change in the process of reproducing the relationships between the pair of objects to be compared and the auxiliary "third." These latter changes, in turn, depended on the degree of difference between the auxiliary objects and the objects in a pair.

This is shown in Tables 7 and 8.

TABLE 7

Degree of Difference Between Shorter Auxiliary Lines and Lines in a Pair

Difference between auxiliary line and line A in percent of line A	Difference between auxiliary line and line A (in mm)			Difference between auxiliary line and line B (in mm)		
	In models	In drawings	Percent of models	In models	In drawings	Percent of models
12	4	5.7	142	15	16.7	111
30	10	9.5	95	21	20.8	99
50	16	13.9	87	27	23.8	88

TABLE 8

Degree of Difference Between Longer Auxiliary Lines and Lines in a Pair

Difference between auxiliary line and line A in percent of line A	Difference between auxiliary line and line A (in mm)			Difference between auxiliary line and line B (in mm)		
	In models	In drawings	Percent of models	In models	In drawings	Percent of models
12	16	18.2	114	5	7.0	140
30	24	25.1	108	13	13.5	104
40	28	27.4	96	17	16.7	98
50	54	47.5	88	43	37.4	87

Tables 7 and 8 show that with a slight difference between the auxiliary line and the lines in the pair,[12] the distinction between the third line and the pair was accentuated in the reproductions, was increased; the pair of lines diverged, as it were, from the third. With a large difference between the auxiliary line and the lines to be compared, the distinction between the third and the pair was smoothed over, and the pair of lines to be compared came close to the third, becoming like it.

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[12] See the first line of Tables 7 and 8.

The experiments showed that for each kind of object there existed a difference between the auxiliary object and the pair to be compared that was neither accentuated nor smoothed over in the drawings, but that corresponded exactly to the size of the relationships in the originals. The introduction of these optimal auxiliary objects resulted in an exact reproduction of the distinction between the objects in a pair, as well as in the precision of absolute and relative dimensions and of the shape of the objects to be compared.

The process of comparison plays a leading role in the distinction or differentiation of objects.

This was revealed quite distinctly in the experiments of Pavlov that the comparison of a definite conditioned stimulus with an agent close to it results incomparably more quickly in the specialization of the conditioned stimulus, in the differential inhibition of external agents, than does the simple, frequent repetition of the same agents. It was established by Pavlov and his pupils that the process of inhibition lies at the basis of differentiation. In experiments in the methods of conditioned reflexes--a fact that is very important for us revealed quite distinctly--it turned out that the intensity of the inhibiting process was determined by the difficulty of the differentiation, or, in other words, the nature of the inhibition depended on the degree of similarity between the stimuli to be differentiated. When a dog was to differentiate a tone and 1/8 of a tone, the inhibition was more intense than when two sounds differing by two tones were differentiated.

These experiments by Belyakov gave Pavlov the right to formulate the following conclusion, which is extremely important for our investigation: "The subtler the differential inhibition, the greater the retardation, and vice versa [3:118]."

Correlating these data with the results of our experiments, we can see that the psychological regularity, which we disclosed experimentally, of the dependence of the results of comparison on the degree of similarity of the objects finds its explanation in the various courses of the physiological process of differential inhibition.

With the introduction of an auxiliary object into comparison, the differential inhibition is also variable, depending on the degree of difference between the auxiliary object and the objects in a pair. When the difference between the pair and the "third" object is very significant, an irradiation of the inhibiting process occurred, which caused a likening of the objects in a pair to a third. With an insignificant, slight difference between the third and the pair, a concentration of the inhibiting process was observed, which intensified the process of differentiating two objects from a third.

Our experiments showed that the mental activity of the comparison of objects in perception is closely related to language, to a person's speech. Moreover, the activity of comparison cannot be properly understood outside the connection of speech, since all relationships established in the comparison-process are formulated verbally by a person.

With the aid of language a person gets to know objects, adding them to a definite category of objects. With the aid of language he establishes and expresses relationships between objects, stating in speech the similarity, difference, or identity (like, unlike, the same, more, less, equal).

Basically, in the comparison process the examinees were faced with the relationships of similarity and difference. In all cases, similarity had a more general nature, and difference a more particular one. These relationships were revealed by the examinees in various ways, depending on how obvious the relationships were and how easy it was to reveal them in comparison.

But in comparing objects in a pair, the examinee often had difficulty determining whether the objects to be compared were like or unlike each other. In these cases he said, "Almost alike," "Almost the same," "It seems similar." After the introduction of an auxiliary object, fully definite relationships were established between the objects in the pair: they were alike or not alike.

Moreover, in some cases of comparison in a pair the children said that the objects were alike, but just how they could not indicate. Introducing an auxiliary object also assisted them in this respect; the children determined how objects were alike and how they were unalike.

Relationships are formulated differently, also, depending on the complexity of the objects to be compared. Thus, the differences between lines were manifested only with respect to length and were expressed in the words: "more, less, equal, longer, shorter." The distinctions between rectangles were established in two respects--width and height. The examinees said that rectangles were "wider," "narrower," "lower," "higher," "a little wider," "a little narrower," "thin," "fat," "long," "broad."

In the comparison of pitchers still more diverse relationships of similarity and difference were established, not only in the dimensions of width and height, but in the general shape of the pitchers and in the shape of the necks, the bottoms, and the handles.

Finally, leaves were compared not only according to dimensions and the general and particular features of shape (outline, teeth, veins, apices, twig-grafts), but according to color as well. Comparison on the basis of color, however, determining in this way whether a tree was old or young and how long ago the leaf had dried, was accessible only to adults; children, as a rule, paid no attention to this.

Thus, the more complex the objects, the more complex were the relationships between their individual features. So, if the children said definitely whether lines were alike or not, in the comparison of more complex objects, such as pitchers or leaves, they noted that the objects were now similar, now different. The experiments showed that in this respect the introduction of an auxiliary object is very necessary, since it helps in singling out and systematizing the signs of both similarity and difference.

An analysis of the statements showed that the children, as a rule, did not try to measure lines mentally, or to correlate them with a measurement scale, as adults did. They determined the relationship between lines only in the most general form: "The first line was a little shorter" (subject E.), or "The second stick was longer, and the first shorter" (subject A.). Thus the relationship between lines was often reproduced incorrectly in words: they said that the "first line was longer than the second," while the reverse relationship occurred in the originals and in the reproductions. These data confirm the observations of Ivanov-Smolensk, who, in his investigations of the interaction of the first and second signal systems in younger children, established that

> ...at a younger age, in a verbal account, in the majority of cases, signals and reactions are described properly, but are reproduced much worse, and frequently the relations, the ties between them are confused: the direct ties, associations (positive and inhibitory) were formed, but they are not yet in a proper verbal skill [2:574].
>
> According to the degree of the child's age-evolution, everything that happens in the first signal system (the bearer of figurative thought) finds an increasingly complete and exact reflection in the second signal system; direct experience (which is imprinted by the first signal system) becomes increasingly accessible, in the expression of I. P. Pavlov, to 'abstraction and generalization,' to verbal interpretation and clear awareness...[2:579].

For comparison and accuracy of the results of the experiments, we conducted an experiment with 15 adults, based on the same method as with the children in third grade. In our experiments the adults used various special devices for remembering the absolute dimensions of the lines. The majority of adults remembered the dimensions of lines in centimeters. Many correlated the measurement of a line with the dimensions of a sheet of paper. In comparing lines in a pair, adults indicated not only the general relationship between the lines, as the children had done, but tried to determine exactly by how much the second line was larger than the first. "The second line (B) is longer, the first is a little shorter. The difference between them is about 1 cm" (subject Sh.).

When a third line was included in the comparison, both adults and children, in their remarks, determined its position with respect to a line in the pair: "The first line was the longest, the second (A) was the shortest, the third (B) was longer than the second" (subject U.). But in contrast to the children, the adults determined that in some cases the size of the auxiliary line was closer to the line A, and in other cases it was closer to the line B. "The second line (B) is almost the same, but slightly shorter" (subject Sh.).

Thus, the experiments showed that with the necessity to determine just how and by how much the objects to be compared differed, the children's weakness was revealed, since they had trouble formulating the relationships between objects and interpreting them on the basis of verbal thinking. Adults, using the generalized forms of second signal ties, reflected more adequately both the individual properties of objects and the relationships between objects when comparing them in a pair, while for third graders the introduction of an auxiliary object was required for precision in reproducing objects to be compared.

General Conclusions

1. The introduction of an auxiliary object exerts an influence on the comparison of objects in a pair. This influence is determined: a) by the degree of similarity of the objects to be compared in a pair; b) by the degree of difference between an auxiliary object and a definite (first or second) member of a pair; c) by the complexity of the objects to be compared.

2. Bringing into comparison an auxiliary object that differs in a definite respect from the objects in a pair promotes a more precise reflection of the distinctiveness of the objects to be compared, in the concepts of third graders. In this case we obtained a more complete singling out of the signs of similarity and difference in the examinees' reproductions than had happened during the comparison of objects in a pair.

3. After the comparison with an auxiliary object, there are exposed, in the main, the signs of similarity and difference between the two objects that appear as a result of establishing more precise relationships with the respective third object.

4. The comparison of visually perceived objects has its basic interaction with the two signal systems. The more complete forms of

this tie in the mental activity of adults aided a more adequate reflection of the objects in the examinees' concepts.

In the experiments it was revealed quite expressively that the result of activity in comparison (the hidden relationships of objects) is always expressed verbally. The more highly developed vocal thought of adults promoted a more complete realization of the comparison process. In having a variety of generalized grammatical forms of speech, adults expressed more precisely in words the relationships between objects while comparing them in a pair, whereas the introduction of a third object was required for third graders' precision, in the concepts, of the objects to be compared.

5. The device we have investigated, of comparing the objects in a pair with a specially selected auxiliary object, can be, in our opinion, applied usefully in teaching various disciplines (for example, natural science and drawing in elementary school, zoology and literature in secondary school). The pedagogical value of the device, from our point of view, is that its application helps the teacher to develop the pupils' thinking, since, in a comparison organized and directed by the teacher, with an appropriate image, the pupils reveal distinctly the most characteristic signs of objects and display the necessary ties and relationships between objects to be compared.

In this comparison by means of exposing signs of similarity and difference in the objects to be compared, concepts that adequately reflect reality are formed in the pupils.

It should be particularly emphasized that these comparisons contribute to the mastery of a system of knowledge.

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ON THE FORMATION OF AN ELEMENTARY CONCEPT OF NUMBER BY THE CHILD

V. V. Davydov[*]

The subject of the experimental investigation reported here was the child's formation of the concept of number, a process that occurs when addition is learned as a mental operation. In the psychology and methodology of arithmetic, it is a known fact that at a certain stage in the development of their knowledge of arithmetic, children are unable to operate with numbers divorced from quantities of objects; arithmetical addition is carried out only with objects, even though the children already know of numbers that establish the size of addends. At subsequent stages of instruction, the children go on to operating with numbers abstracted from objects. This transition is an objective process within which the conceptual form of the number is developed.

The investigation established consecutive stages in this process:

1) adding quantities of things by counting the units (objective operation method);

2) adding abstract quantities by counting the units (detailed verbal operation method);

3) adding abstract quantities by counting the second addend onto the first taken as a whole (conceptual operation method proper).

The characteristic feature of the object stage of performing addition was that when given a verbal assignment, the children would make up things to use as objects in the proposed addition (first operation) and then find the result by counting them in succession (second operation). It was shown that at this stage, a method of operating such as counting the addends by units starting with one was perfectly legitimate, even though the children knew the numerical designation of an addend and could relate a numeral to the whole quantity to be designated. Here their knowledge of the size of an aggregate of objects was not used as a simple integral description of the quantity.

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If the children were given the first addend in the form of a digit, they took the digit for a unit, for one object, regardless of what it designated. These children still did not conceive of a definite abstract quantity behind the figure. They were first to make up a real quantity based on the numeral and then use it in adding. The children were able to think up a group of arbitrary objects to correspond exactly to the given numeral. That is to say, to them a numeral was a completely generalized and differentiated indication of a collection of things. Thus, at the object stage of adding, children know a numeral as an indication, the content of which can be shown to them when a series of names are actually related to objects, in detail.

The distinctive feature of the second stage of addition (detailed verbal counting) was that the children no longer related a specific designation to an immediate collection of things. An assignment would be carried out without regard to objects, but counting was retained as the method of adding (the problem "3 + 2" was done like this: "1, 2, 3--4, 5"; the dash signifies the changeover to the second addend). When the addend was given as a numeral, it was not taken for a unit, but prompted verbal counting accompanied by tapping the numeral with a finger.

The investigation established that this stage of addition, under the experimental conditions, originated due to the fact that, though the objects were hidden, their actual presence was emphasized verbally, and they were shown to the child from time to time. Under such conditions, the children would begin to make particular movements (such as pressing a finger against the box in which the objects were hidden) while pronouncing the number series from one to the specified designation (counting).

Characteristically, the movements (and therefore the verbal counting) were first seen only when the hidden objects were actually present. As soon as they were removed entirely, the children refused to perform addition. Addition ceased whenever the experimenter forbade them to make the movements. That reciting the numbers while counting depended on the hand movements was shown in this fact: When the children began to count verbally, it was distinctly noted that the numbers were named after the movements.

Gradually the children progressed to assignments presented in words without using objects. True, at first a special, verbal indication of the supposed presence of objects was required (in naming the first addend, the experimenter would say "Here's 5" — a gesture in the direction of the table -- "and add 3").

After numerous problems given without objects had been performed, the character of the child's hand movements was altered. Slow pressures became rapid tapping of the finger on the table, and this was in turn replaced by an accentuated recitation of the numbers while counting. Thus the movements, done on such a large scale at first, were gradually curtailed or reduced, so that as a result, the child would do a verbal problem by counting purely in words (for example, the problem "5 + 3" was done like this: "1,2,3,4,5 -- 6,7,8; the answer is 8").

There is reason to believe that the movements, even when quite reduced, never completely disappear. Hand movements are replaced by the accentuated pronunciation of numbers. As our findings showed, the pronunciation of numerals in adding had different intonations than in simple counting.

An analysis of how the movements we have described originate shows the following. Back at the object addition stage, when quantities of things were being added, the children would touch the objects with their fingers while counting. When forced to operate with a quantity not presented in material form, they apparently used this system of movements acquired while adding real objects. These movements were first repeated, not in the complete absence of objects, but only when the objects were hidden, though still unavailable. One can theorize that hand movements let the children, so to speak, simultaneously re-establish and use each element of the hidden (and subsequently totally absent) quantity as objects in the operation. In this case the operation is effected as if real objects were present that could be used in adding. But since the child has no articles he can use as immediate objects in the operation, one might say that when doing addition verbally he "implies" a definite quantity in the numeral itself. This implication is manifested in the form of the direct coincidence of two factors -- the special hand movements toward the hidden objects (re-establishing them as objects in the operation) and the use of them in the adding process.

Gradually the link between the movements and the presence of hidden objects weakens, and the movements themselves are reduced -- the children learn how to carry out an oral assignment in a purely verbal way. But this is no longer operating with words, as often happens at the earliest stage of learning how to count; it is operating with a real, though implied, quantity. When objects were being added, a special procedure for constructing an object of the operation, the object addend, was carried out beforehand. Now this procedure drops out of the operation (is curtailed), for the children have learned to imply a quantity in a given numeral.

Having learned how to add without objects, they progress to adding abstract quantities. Indeed, now they are operating with numbers as such, without expressing them in any concrete, object form. We suppose that the special hand movements (their reduced form is the accentuated pronunciation of the numbers) related to absent objects are a mechanism for implying them.

From this standpoint, the question of the seeming irrelevance of the operation to immediate objects when addition is done purely in words can be resolved. Behind this irrelevance to immediate objects are hidden real mechanisms (a reduced form of the special movements) that actually allow the children to effect a relation between the quantity and the real one, which is the genuine subject of the operation and is its ultimate determinant. Moreover, it can be said that only in this relating process is the numeral thought of as a number (quantity), when addition is being done without objects.

When children begin to think of quantities as implied by numerals, counting is still necessary, for it is the real act that creates the implication and ensures use of all the elements of a quantity. While the operation is being reinforced in the realm of abstract quantities, it is not necessary to pass through all the elements of the first addend. When a quite definite quantity is implied in the numeral itself, which is known in advance, there is the possibility of skipping the middle elements of the series being counted. This circumstance eliminates the necessity of counting the first addend and thereby teaches the child how to use a number as a whole. It is the latter that characterizes use of numbers in concept form.

Outwardly, the curtailment of counting looks like a transition to adding on, which immediately seems to be simply counting "farther" than the first addend ("5 + 3" -- "5 -- 6,7,8; the answer is 8"). Could it be that curtailment of counting begins when the children learn to count farther from any given number? Special experiments have shown that the child who has learned to add on in this form cannot use this skill when adding objects, which he can do only by counting. Such instruction in adding-on creates only an imaginary concept, since the numeral is not related to an object quantity as a whole.

Observation of the methods of adding in children who thoroughly understand the number concept has proven that at the moment they name one quantity and add another to it, they make a continuous movement of the hand along the objects of the first quantity. This movement is accompanied by a protraction of the sound of the number (for example, for the object problem "6 + 3," a child would move his hand along all the objects for the first addend without stopping, say the number "si-i-i-x," and go on to add the elements of the second addend to it -- "7, 8, 9; the answer is 9").

Guided by these observations, we experimentally produced curtailment of counting by having the child make a continuous movement along the objects for a designated quantity. This movement, so to speak, reconstructed the designated series, but very quickly: Real, discrete counting, with pauses at every concrete designation, was not produced. In distinction from actual counting, the movement made along the series of objects can be called conditional counting, which allows a child to use all the units of a quantity without relating a numeral to each one separately.

Gradually the continuous movement is transformed into a simple gesture directed at the group of objects and accompanied by an accentuated pronunciation of the number. Finally, even the gesture may be reduced. There remains only the simple verbal designation of the first addend and the addition of the elements of the second to it. The child has learned how to operate with numbers. But if this child is told to carry out addition with objects again, he will do it by adding on, in the form of an extended but continuous movement along the objects. This action shows ability to take the first addend as a whole, which corresponds to the conceptual form of a number.

In the investigation of the gradual formation of the concept, the question may arise, why not teach at first the operation in the form that characterizes the highest level? The experimental and theoretical analysis we performed showed that an operation that is created without gradual transformations is a sham, since the child who is doing it is not aware (does not understand) the nature of the conversions that occur in it. He cannot correlate the components of his operation and its result with the objective reality that is the ultimate determinant of the mental operations of arithmetic.

On the other hand, the gradual formation of a mental operation and its object -- the number concept -- allows a child to be taught to correlate an abstract transformation with its primary source -- an object transformation. Then the meaning of the reality behind an abstract number becomes clear, and the operations are performed that actually determine the result of the arithmetical transformation.

THE GENERALIZED CONCEPTION IN PROBLEM SOLVING

A. V. Brushlinskii*

The fundamental purpose of psychological research into thinking is to study thought as a process (analysis, synthesis, and generalization) included in the specific activity of an individual [6]. This means primarily that underlying the external results (products, formations) of thought is a thought process, appearing directly but leading to those results, and this thought process must be conjectured.

One of the components of the thought process is the maximally generalized conception of the solution of a problem [1]. Considering the results of investigations in which the role of conception was studied in one form or another in the solution of problems [2, 3, 5, 8], we planned to investigate how the initial, maximally generalized conception is used and developed in a subject's subsequent mental process. Such a conception is effected in the course of analysis, synthesis, and generalization of all the conditions and requirements of a problem. Unless these processes are revealed it is impossible to explain correctly why a conception leads to the necessary result--the solution of the given problem.

For example, in certain of Selz's works (see especially [7]), it inevitably remains unexplained how this "filling in" of an anticipating scheme for the solution proceeds, because, from his point of view, the content of a problem, i.e., the goal of the cognitive process, is in general not included in thought expressed through operations purely externally connected with the problem. Consequently, there can be no discussion of the process of analyzing the problem by the subject himself (this is discussed in more detail in [6]).

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Methodology

The subjects (adults and pupils--68 persons in all) were given a problem from the school geometry course. During the solution, different groups of subjects received appropriate prompting (for example, the theorem on which the solution is based). By examining the subjects' utilization of the promptings, it was possible to study objectively the thought process of the individual developing his conception. For purposes of comparison, some subjects were required to solve the problem without promptings. A problem was presented in a formulation similar to the following example.

> Point M is taken inside triangle ABC (see Figure 1) and parallelograms $AMBM_1$, $BMCM_2$, and $CMAM_3$ are constructed. Prove that straight lines AM_2, BM_3, and CM_1 intersect at one point.

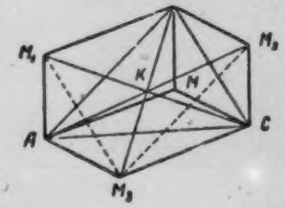

Figure 1

Solution. In parallelogram $BMCM_2$, side BM_2 is equal and parallel to side MC; in parallelogram $CMAM_3$, side AM_3 is equal and parallel to side MC, and hence side BM_2 in quadrilateral AM_3M_2B is equal and parallel to side AM_3; consequently, AM_3M_2B is a parallelogram, hence its diagonals AM_2 and BM_3 at their point of intersection (K) are bisected. Analogously it is proved that in parallelogram CBM_1M_3, diagonal CM_1 intersects diagonal BM_3 at its midpoint K. Therefore lines AM_2, BM_3, and CM_1 intersect at one point. Q. E. D. (In place of one of the two parallelograms AM_3M_2B and CBM_1M_3, one may also select ACM_2M_1.)

Thus, to solve the problem it is necessary to construct supplementary lines, say M_2M_3 and M_3M_1, to use two parallelograms (of three possible ones) not given in the conditions, and to prove the intersection of three lines in one point, not directly, but by considering the bisection of the diagonals. Here one must enlist the theorem of the diagonals of a parallelogram being bisected at their point of intersection.

With one group of subjects (who were shown the problem beforehand) we ascertained the extent of their knowledge of geometry. They were to recall (sometimes with the experimenter's help) a series of theorems quite unrelated to the problem, as well as the theorem of the diagonals of a parallelogram being bisected at their point of intersection. Thus the possible proposition that a subject simply "forgot" a necessary theorem and was thus unable to use it in a solution was repudiated from the start.

In subsequent series the experimenter prompted by stating the theorem on which a solution was based or by directly indicating the parallelogram containing intersecting segments as diagonals.

Results

First we shall present the results of the experiments of the series in which the subject's knowledge of geometry was checked in advance. In these experiments the subjects recalled (in response to the experimenter's question) the properties of the intersection of parallel lines, the properties of the rhombus, of chords, etc., and even the properties of the diagonals of a parallelogram. Then a problem was given them. As they began to analyze it, the subjects formulated their general conception quite precisely: To find some figures within which certain segments intersect at one point. This conception originated as a unique generalization of the repeated past use of several theorems on "the intersection of segments," when, to prove the intersection of lines, one had to regard them as elements within a particular figure, e.g., as the angle bisectors of a triangle intersecting at one point or as the diagonals of several polygons.

Let us consider in detail the typical course of solution with the example of subject V. S., using the data from other experiments for comparison.

At the first stage of the thought process, the subject, as a result of his initial analysis of the problem, formulated a maximally generalized conception: "One must look for some figures, such as triangles, perhaps larger than here, in which lines would intersect."

Other subjects composed the conception of the solution similarly: "If we prove that intersecting lines are segments within some figure, then their point of intersection is, say, the center of a circumscribed circle," or "What are these lines with respect to the given figures? We must find a figure that would embrace all three lines."

These are representations of the maximally generalized conception of the solution that originates at the first stage. It is so general that it does not even formulate a principle of solution (such as a general proposition or a theorem).

The conception is directed at ascertaining some specified (although in only the most general terms) figure (or figures). Owing to its extreme generality, however, it does not inherently contain an indication of the isolation of precisely that figure required for the solution. To break it down is possible only on the basis of further analysis of the problem, to make the initial conception more concrete.

At the next (second) stage of the thought process the search for a figure containing intersecting segments began. Subject V. S. first isolated triangle ABC as such a figure (other subjects considered the same triangle, or parallelograms given in the conditions, hexagon $AM_1BM_2CM_3$, etc.). Here he analyzed and attempted to apply the theorem of the intersection of the angle bisectors of the triangle in one point (on actualization, see [9]). That is, he considered directly only the intersection of the segments, completely overlooking their bisection at the point of intersection, which was required for the solution. Gradually he became convinced that the intersecting segments are not bisectors, since, in his words, "point M can be anywhere within triangle ABC and point K will move correspondingly to any position" (see Figure 1).

With this, the second stage of the thought process is concluded. The initial attempt to make concrete the maximally generalized schematic conception in which particular conditions (concrete figures) are represented only as altered (abstract, more precisely, not yet specified) values ("some" figures) is done. As the first particular value, a triangle with bisectors is subsumed into this general scheme.

After an unsuccessful attempt, V. S. made a second attempt to make the conception concrete: He began to seek another figure that would satisfy the problem's requirements (the third stage of the mental process). Thus, the transition from one stage of the thought process to another retains a unique transitional determination of the entire thought process, insofar as that same very general conception is realized now in a new particular variant.

At the third stage the subject examined several figures in succession very rapidly and superficially. He did not analyze any one of them in detail, since he did not know "what this yields" toward the realization of the conception (in the course of the thought process only an analysis of objects, which can somehow be correlated with the requirements of the problem through synthesis, is possible).

"Line AM_2," the subject remarked, "connects one of the triangle's sides with the opposite vertex of a parallelogram. Figure ACM_2B has two triangles, ABC and BCM_2. AM_2 passes through them. It would be possible to ascertain the equality of the angles and sides of these angles, but what will this yield?"

As is evident from the records of the experiment, the subject visually selected individual figures and segments in the drawing. His analysis (and synthesis) of these geometric elements was predominantly sensory, still insufficiently linked in a single process with strictly mental analysis and generalization (for example, when using theorems). He viewed hexagon $AM_1BM_2CM_3$ in the same way: "We can ascertain something from the six triangles at the common vertex K." Here he noticed the property, which is necessary for the solution, of the intersecting segments' bisection. "Purely intuitively I saw that point K divides these three lines in half, but I am still not sure this is so. . . I still don't see any way . . . ." He seemed to suggest to himself the essential feature of the given segments, but he did not attempt to check his assumption and made no use of his "self-prompting." The analysis of the problem was still not advanced enough for him to "accept" it. Directing the course of solution was a generalized conception in which the fact that the segments are bisected was not taken into direct consideration. Hence the subject did not take into account the division of the intersecting segments even though he had noticed it.

During further, basically sensory analysis of the problem, triangles AM_2C, AM_3B, and AM_1C were selected. "Each of these triangles has a side of triangle ABC, one side of a parallelogram, and one of the intersecting lines." The subject began to consider these in detail, gradually making the transition to strictly mental analysis (the beginning of the fourth stage of the thought process; see below).

Thus, attempts to make the conception concrete are continued at the third stage. But not one of the figures isolated satisfied the subject. Essentially, the conception was not given a definite, detailed concretization here, for at the third stage, as noted above, the analysis (and synthesis) of the problem was chiefly sensory (visual). Particular properties of objects (without using any theorems to establish the general features of geometric elements) were examined. If only this kind of analysis, which is insufficiently included in the strictly mental process is used, the given problem cannot be solved.

It is clearly apparent at the third stage that at a certain point in its realization, the generalized conception is still unconnected with the theorems and does not include them in its composition (until they are actualized). As noted above, they do not enter into its content even when it is formulated at the start of the solution. Consequently, no conception in general occurs here except for the maximally generalized one.

Not every solution is begun with the composition of a general conception [9]. Moreover, the conception is not always the formulation of a principle (general statement, theorem, etc.). In the case in question it appears in the form of a very general scheme of operations, geometric constructions (i.e., in the form of an operational scheme) through which one may connect elements contained in the conditions and requirements of the problem. Only then can one correlate them and express their correlation through some geometr proposition (a theorem, for example). Only the general basic relationship of objects (the intersection of lines) is considered in the conception. This is a unique *method of* initial *generalization* of the essential connections and interdependencies of the problem and is basic to the exposition of the subsequent thought process. The vague values ("some figures") of this scheme are gradually replaced by different particular propositions (concrete figures)

The next attempt at making the conception concrete was undertaken at the *fourth stage* of the thought process. V. S. began to examine closely the sides and angles of figures AM_2C, AM_3B and AM_1C, actualizing several theorems in which properties of parallelograms, angles with correspondingly parallel sides, etc., were generalized. He ascertained the equality of segments AM_3, MC, and BM_2, as well as the equality of the sides of other parallelograms. Then he examined the two triangles ABM_3 and ABM_2 very closely "Their common side is AB. . . . Since AM_3 and BM_2 are equal and parallel, their angles are equal" (he isolates angles BM_2A and AM_3B, ABM_2 and BAM_3, and many others).[1] As a result, the subject draws "supplementary line M_2M_3" and analyzes the parallelogram ABM_2M_3 thus formed, correlating it with the conception (this is the most interesting point in the solution). The theorem, however, is reproduced in a unique formulation: "The diagonals of a parallelogram *intersect* at one point and are *bisected*." (The correct statement reads:

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[1] Now one may predict that if in the future the subject picks out one parallelogram of the three that are possible and necessary for the solution, this one will be parallelogram ABM_2M_3, which seems to be composed of the two triangles already analyzed, ABM_3 and ABM_2.

"If a quadrangle is a parallelogram, then its intersecting diagonals are bisected.")

The theorem is thereby broken down into two seemingly equivalent "parts" (intersection and bisection), but since the analysis of the problem, according to the conception, is directly aimed at isolating only the features of the intersection of segments, only this first "part" is included into the thought process. Then the subject picks another parallelogram (AM_1M_2C) in which "the diagonals also intersect at one point." "I think this is the solution," he says, not even mentioning the bisection.

Thus, at this stage of analysis of the problem the second "part" of the theorem, the statement concerning the bisection of the diagonals, is excluded ("thrown out") from the thought process. At first subject V. S. (as well as the others) assumed that for the solution it is sufficient to be convinced of the intersection of the diagonals of each of the two parallelograms. He still did not see the need to prove the coincidence of the points of intersection of the diagonals of both parallelograms (for which one must use the feature of bisection at these points). The real, objective role of the generalized conception comes forth most distinctly in this fact. Formulated at the first stage of the solution, this conception expresses that level of analysis of the problem at which only the properties of the intersection of segments within "some" figures are essential for the subject. Determining correspondingly the general direction of the subsequent thought process (at first directly with the intersection of lines), the conception becomes a means for further analysis of the problem (on the means of analysis, see [4]).

Therefore at first glance there seems to be a paradox. The two parallelograms and their diagonals, objectively necessary for the solution, have already been ascertained, and the necessary theorem directly indicating the bisection of the segments has been actualized, but nevertheless their division is still overlooked and the main content of the theorem is discarded as something inessential. The new data, geometrically quite pertinent, still have no value psychologically (for the subject) in the course of the solution.

With the aid of the analysis thus conducted, the subject, as we saw, separated a series of essential properties of objects (parallelograms, diagonals, etc.) that were necessary, but insufficient, for the solution.

As he continued to examine the intersection of the three segments, the subject also began to understand that a direct proof that the three lines intersect at one point--without considering the division into halves--was impossible (the fifth stage of the thought process).

We cite a relevant excerpt from the record:

> Here are two parallelograms. The diagonals are AM_2 and M_3B, intersecting at point K. Diagonal AM_2--again the same one--and diagonal M_1C, they too intersect at some point K_1. Now we need only prove that both points coincide. . . [after a pause the other "part" of the theorem, the bisecting, is secondarily analyzed]. But indeed the diagonals are bisected into halves by the intersection [this then leads to the solution].

At this fifth stage the content of the conception was reorganized. The subject acquired a new goal--to prove that points K and K_1 coincide--and took into consideration the bisection of the diagonals. As a result he solved the problem.

Discussion

Thus, using a typical problem, we have briefly examined, how thought proceeds as it leads to the solution, and we have observed its fundamental stages.

The first stage is formulating a conception; the second is making the conception concrete; the third is a sensory analysis of predominantly particular properties of geometric objects (creation of conditions for later concretization); the fourth stage is the second concretization; and the fifth is alteration of the conception and the solution. Approximately the same stages were noted in other subjects, but they were expressed somewhat differently (for example, there could be three or even four detailed concretizations of the conception, a less developed stage of visual analysis, etc.).

Thus, in the case in question, the thought process is a realization of a maximally generalized conception to direct the course of solution. The direction (selectiveness) of the thought process of a person who is realizing a generalized conception of a solution becomes concrete principally when at a specific stage in the analysis of a problem, the subject first considers directly only the intersection of the segments, overlooking completely their bisection, even if he himself selects the features of this bisection.

This basic fact appeared distinctly, in one form or another, in all 16 subjects of this series of experiments. All of them, after their conception, first discarded as inessential the properties of the bisection of the intersecting segments, although they themselves often seemed to "come across" them directly and isolate them.

Let us cite some more examples. Subject N. N. reproduced the necessary theorem in the course of the solution: ". . .in a parallelogram the diagonals are bisected at the point of their intersection," but at first she made no use of it at all. The subject's analysis of the problem was then directed only toward the intersection: "Which lines intersect in the triangle? Which other lines intersect at a point in general, in whatever figure? Perhaps lines intersect at a point in all hexagons?" Subject T. M., about to mention the division of the segments, immediately disregarded this idea: "But what else intersects at one point and is divided into halves? Not divided into halves, but intersects at one point. The altitude of a triangle? The medians?" .

These cited cases, expressing a certain direction of human thought, allow us to conclude that in the cases in question a person's thought process occurs in the form of realization of a maximally generalized conception of the solution which arises at the early stages of the analysis of a problem. Conception of the solution is the device for original generalization of the fundamental relationships of a problem, on the basis of which its subsequent analysis is developed.

This fundamental conclusion also stems from subsequent experiments. In some of them the subjects were prompted (at early stages of the thought process) in the theorem on which the solution was based, or the parallelograms including the intersecting segments were directly indicated. In both cases, after being prompted, the subjects first, according to their conception, examined only the intersection of the diagonals of the parallelograms, completely overlooking their bisection.

Thus, the realization of the generalized conception is one of the forms of the thought process of the individual. The conception of the solution as a means for the further analysis of a problem concretely expresses the definite direction (selectiveness [izbiratel'nost']) of human thought. This helps to explain why, in the course of the solution, the subjects use and analyze some properties of objects but disregard others (excluded, at least temporarily, from the thought process) or do not notice them at all.

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AN ANALYSIS OF THE PROCESS OF SOLVING SIMPLE ARITHMETIC PROBLEMS

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Report I: The Subject and Goals of This Investigation. Indirect Problems

Introduction

The development of modern methods of production makes ever higher demands on the individual. A worker needs to achieve an increasingly high level of education, for the scope of knowledge needed for work is expanding. The constant turnover in industry associated with a change in the professions of many people, demands an increasingly high level of general education. With the present state of our knowledge and methods of teaching, this high level of general education can be attained only over a significantly long period of instruction or by overloading the pupils. Neither condition is practical. Therefore, the solution for this quite acute situation must be sought in different methods.

One solution is to restructure knowledge itself and alter content of school subjects. Knowledge should be "condensed" or reduced, but it should encompass a broader and continually expanding range of objective phenomena. The structure of knowledge should be simplified, and algorithms for its use should be less cumbersome.

Another way of shortening instruction time is through maximal success of the instruction process. Of major import here is the transition to the so-called active methods of instruction and training that would allow the pupils to master the necessary knowledge and skills in the shortest time and with the least effort (see [6]).

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Finally, a third way to solve the problem consists of trying to "pull down" some sections of the school curriculum into preschool instruction-- that is, to use preschool instruction for preparing a specific base that would facilitate assimilation of the school curriculum. This approach is very realistic, and it would be difficult to overestimate its significance.

But implementation of any of these measures is hindered by our lack of knowledge of the structure of human activity-- of thought processes in particular. Therefore, the first condition and prerequisite for all attempts at a practical solution of the problem is the development of a wide range of logical, psychological, and pedagogical investigations into the structure of human activity.

Statement of Theoretical Principles

These considerations determined the aims of this investigation. In selecting the specific empirical material and in outlining the general plan of the work, we began from the following theoretical principles:

Principle 1. The basis of a child's intellectual development is the assimilation of the cultural elements accumulated by mankind, and mastery of socially developed knowledge and methods of activity that confront him in the form of work production, language (understood broadly as an aggregate of symbolic systems), and everyday practice in the environment (see [4]).

Principle 2. In view of the foregoing, everyone's knowledge and methods of activity (including mental operations) must be considered on two levels that, though interrelated, are nevertheless essentially different.

> a) In a standard composition and structure, which alone can facilitate the solution of specific problems: In this respect they appear as a "work norm" and do not depend on the subjective means of individual persons. This is what is assimilated, what is mastered.

b) From the standpoint of the operations that persons can and should perform so that, beginning from specific, previously mastered information and methods of operating, they can master a new complex of knowledge and operations, a new "norm" (for more details on this, see [4]).

Principle 3. Knowledge and methods of operation (including mental operations) are mastered only in a definite system: Any information and mental operations can be mastered only on the basis of others, and they in turn form conditions and prerequisites for the mastery of still other more complex information and operations. The result is that during instruction, information and mental operations form a single system in which all elements are mutually connected and interdependent, each preceding "layer" determining the character of the next, and all of them as a whole depending on what requirements we make for instruction in its entirety.

From this last principle it follows, in particular that preschool education must not be viewed in isolation; it is the initial element in the entire educational system and hence should be considered in relation to other subsequent elements, especially the system for teaching primary school children. In other words, preschool instruction should be regarded as a preparation for teaching primary school children. In particular, the content of preschool instruction is directly determined by the content of elementary school instruction.

Therefore, to determine the content of preschool instruction, at least in a narrow area, we had to begin with an analysis "from above"--an analysis of what this preschool instruction prepares the child for. We singled out processes of solving arithmetic problems in the first grade, assuming that these processes are included in a type of "synthetic" mental activity, a concentration of much of the abilities, skills, and information that the child must master in the preschool period.

We had to analyze the processes of solving arithemtic problems so as to single out not only the structure and composition of the completed activity, but also the information and mental operations prerequisite for "putting it together," i.e., mastering it. This was the first problem. The second problem was defining the subordination and coordination of all knowledge and operations revealed during the analysis. It thus outlined (approximately) the sequence of the relevant study material. Later, a third problem was to be presented: to determine the structure of that "subjective" activity of the children by which they master socially fixed knowledge and methods of operating the "norm." The fourth problem, which arose after the first three, consisted of investigating the educator's activity in teaching all this information and mental operations. The resolution of these four problems would permit construction of reasonable and effective methodologies of preschool instruction, taking into account logical, psychological, and didactic factors in instruction and training.

Observing Problem Solving by Children

First and often second graders have great difficulty with problems in which the process described by a material situation seems to "diverge" in content or "sense" from the operation that must be done with numbers to obtain the solution. Consider, as an example, a situation in which certain number of things were obtained by combining two sets but it was required to find the number of one of these sets, and subtraction was to be used. Or conversely: Suppose one separated or singled out a part of a set of objects and the student was asked to find the number of the original set by using addition.

We decided to give special attention to such problems, since analyzing them would undoubtedly help to explain the peculiarities of the standard methods of solution as well as shortcomings in the instruction. For a whole series of children, answers like these were typical:

Serezha B., second grade, October

Experimenter: Someone took 6 buckets of water from a barrel, and 9 buckets were left. How many buckets of water had been in the barrel?

Serezha: How many did they take out?

E.: 6

P.: [whispers] 9 and 6 . . . I don't get it . . .There were 3 buckets, right?

Valerik Kh., second grade, September

Experimenter: There were 14 balls in a kindergarten. Ten of them were black, and the others were white. How many white balls were there?

Valerik: [after reading the problem] It's clear that 14 + 10 = 24. Correct?

We notice immediately that these same pupils have no trouble solving problems in which there is no "disparity" between the "sense" of the processes in a material situation and the "sense" of arithmetical operations. For example, they can solve those problems in which a part is separated from a total quantity and one must subtract to find the number of the remaining part, or when two groups are combined and one must add to find the number of the whole set.

One may conclude from this that the cause of difficulties with problems of the type indicated above is not that the arithmetical operations of addition and subtraction are not mastered, nor is it that these operations are just formally mastered, without understanding. At any rate, if these operations are not mastered or understood, it is only in problems of the type indicated above that the deficiency is noticeable.

Views of Methodologists and Psychologists

The pupils' difficulties in solving such problems have interested educators and psychologists for a long time. These problems have even been given a special name: indirect problems.

Galanin [2] discusses especially those difficulties that children may have with problems which require finding the "unknown item" by subtraction. His explanation for the difficulties is that in problems on finding the "unknown item" there is no word (!) that could be replaced by the minus sign. Hence this sign must be positioned by the pupil "according to the sense of the problem" or, as Galanin writes, "according to the definition of the operation as the opposite of addition [2: 64]."

To clarify this explanation and, in general, the whole course of Galanin's thinking, one must set forth his interpretation of pupils' activity in solving ordinary, not indirect, problems. Several paragraphs earlier, considering instruction in "the concepts of addition and subtraction," Galanin writes that to solve direct problems one must subsume the verbal expressions denoting changes in sets of objects ("became," "obtained," "poured out," "won") under one of the mathematical concepts --"addition" or "increase," and "subtraction" or "decrease"--and must associate the designation of this concept with the corresponding mathematical sign [2: 58-59].

The ability to solve problems, from the standpoint of this interpretation, is the result of inductive generalization of the meaning of different verbal expressions denoting a change in the relationships between parts of object groups (or operations entailing such changes). Correspondingly, the teacher's work should consist in helping the children, through appropriate selection of problems and indications of the similarity of various operations (from the standpoint of whether they lead to a decrease or an increase of the original quantity), to complete this generalization and thereby master a particular device for solving problems.

It is quite obvious that solving indirect problems in this way is impossible--incidentally, just as it is impossible to solve all other problems in which there are no operations of increasing or reducing the original quantity and no words designating them. Then there appears this portentous statement to the effect that indirect problems must be solved on another basis, that the sign, and correspondingly, the mathematical operation in indirect problems should be chosen "according to the sense of the problem." But one may ask what the "sense of the problem" is. On what is it based? What must the child know and understand in order to grasp the "sense" of an indirect problem?

In Galanin's opinion, indirect problems should be solved on the basis of understanding specific mathematical relationships. He write that these problems must be explained "in order to create in the pupil a conception of the fact that he is given one quantity and the sum of that quantity and another, to obtain the other he must subtract the first quantity from the sum. From this comes the second of our comments given above, which states that in such problems subtraction is defined as the operation which is "the opposite of addition [2: 64]."

In connection with the plan of the further analysis we want to take special note of three points in Galanin's conception.

First. In analyzing the process of solving ordinary, nonindirect problems, Galanin says nothing of understanding. There the entire instruction process is apparently build on the development of particular associations, and the process of solving a problem appears to be the applicaiton of these associations.

Second. Understanding, required for solving indirect problems, is characterized by Galanin only from the standpoint of content (one must know that the sum of two quantities and one of the quantities are given): he says nothing of the mechanism of this understanding and does not show how one is to teach this understanding.

Third. For solving direct and indirect problems Galanin proposes two different methods. But if the first method he proposes has such narrow application and is inapplicable in solving indirect problems, they perhaps it is not a real method at all. Perhaps it is completely erroneous, and one must seek another method that would be applicable to solving all arithmetic problems.

Kavun and Popova [3] develop the interpretation of the mechanism of the child's activity, which Galanin only skimmed over, quite distinctly and sharply. The authors maintain forthrightly that in arithmetic problems an operation and the solution are chosen on the basis of the creation of "an association between the terms add and subtract and those diverse expressions characterizing addition and subtraction in problems [3]. Their proposed methodology of instruction is built, of course, around this principle.

Skatkin [10] also gives special attention to the problems we are interested in and stresses their difficulty for children. In his classification of simple problems he calls them "problems expressed in indirect form," or "mutually inverse" with respect to simple problems on finding the sum or the difference.

In solving simple problems, the operation is chosen, in Skatkin's opinion, "on the basis of the pupil's life experience, by analogy to the way in which he learned how many objects were obtained when several objects were to be added or taken away [4: 12]." In solving indirect problems, the necessary operation, on the other hand, is found by reasoning, which permits deep penetration into the sense of the problem and, on this basis, its solution. The cause of incorrect solution of problems, accordingly, is the pupils' inability to reason and penetrate the sense of the problem.

If we try to imagine that theoretical interpretation of the child's activity in solving problems from which one may advance such statements, we must admit that it coincides essentially with Galanin's theoretical interpretation, differing from it only by being less clear and complete. True, Skatkin apparently recognizes the insufficiency of this interpretation. In particular, he criticizes the statement, quoted above from the methodology of Kavun and Popova, correctly noting that it is the use of the above mentioned association that leads to mistakes by the children in solving problems expressed indirectly. But he does not deny this principle altogether; he does not say that the problem-solving mechanism should be essentially different. He accepts it in general, believing that it need only be supplemented by the children's "deep penetration" into the sense of the problem.

Finally, like Galanin, Skatkin considers an understanding of the "sense" of indirect problems a necessary condition for solving them, but he remains quite vague on: a) what he means by the sense of a problem, b) what he means by an understanding of the sense, c) how this understanding can be taught.

Finally, the thesis that children who solve indirect problems incorrectly do not understand their meaning arouses our doubts from still another angle. As early as 1915, Ern [1] noted this curious fact: In solving problems expressed in indirect form, some children answer correctly, but write out the solution incorrectly. Ern himself explained this by saying that the pupils assign too much importance to the "external form" of the problem's text and are not accustomed to think about the problem's "internal sense." It is this, in his opinion, which prevents them from fully understanding the operations of addition and subtraction.

In our opinion, this is a very important observation, but a completely incorrect explanation. It is quite obvious that one cannot obtain a correct answer to a problem without thinking about it and understanding the "internal sense" of its situation. Moreover, since the child solves the problem correctly, we may conclude that he not only understands its sense, but also has a definite method for the solution. That the child here cannot select the arithmetical operation correctly and therefore cannot write out the solution correctly speaks for the existence of some phenomena, more complex than mere lack of understanding of the sense, which requires more careful analysis.

The Role of Understanding

In his remarks, Ern [1] describes a problem in which the "subtrahend" and the "remainder" are given, and the minuend is to be found (by adding). First, we decided to ascertain whether there are similar disparities between the answer and the arithmetical notation of the solution in indirect problems of another type. We also wanted to check whether the inability to solve a problem involved a lack of understanding of the sense of the problem's situation.

Even our first observations here indicated that the incorrect solution of a problem could be completely unrelated to a failure to understand its situation.

For example, second-grader Serezha B., below average in arithmetic was given this problem:

> To decorate a fir tree the first-graders made 20 toys. Six of them were paper and the others were cardboard. How many cardboard toys did they make?

Serezha solved it incorrectly: "20 + 6 = 26." However, subsequent conversation showed that this incorrect solution was by no means a consequence of his not understanding the situation described in the problem.

Experimenter: How many toys did they make?

Serezha: Twenty.

E.: What were they made of?

P.: Cardboard and paper.

E.: How many were made out of paper?

P.: Six

E.: What were the rest made of?

P.: Cardboard.

E.: Which toys were there more of -- all of them together, or only cardboard ones?

P.: There were more of all of them.

E.: How many toys, then, were made of cardboard?

P.: [writes]: 20 + 6 = 26.

Thus, the boy not only knew that the cardboard toys entered into the total number of toys made, but he also understood that the total number of toys was larger than the cardboard ones alone. That is, it would seem that he even understood that the cardboard toys constituted a part of the total made. Nevertheless, he continued to solve the problem incorrectly.

Very many records like these could be cited, all confirming the thesis advanced above. Even more striking are the cases in which the children solve the problem correctly and write out its solution or select the arithmetical operation incorrectly.

In December the first-graders were given this problem:

> "Kolya had to make 8 flags. He made 4 flags. How many flags did he still have to make?"
>
> The problem was read twice, and then three children recounted its situation to the class. The teacher asked how many flags Kolya still had to make. The next question, asked only of the above-average pupils, was: "How can we find out how many flags remained for Kolya to make?" The answers were:
>
> Vitya K.: Add 4 to 4
>
> Lena F.: Add 4 to 8.

Sasha S.: Add 4 to 4.

Ira O.: The number 8 consists of 4 and 4.

Tolya B: Add 4 units to 4 units.

Alesha L.: To 4 add 4 more, giving the correct answer, 8.

Tanya S: He made 4, he still had 4 to make.

Vera K.: To 4 units add 8.

Gena Z.: 8 take away 4.

That these children's answers are by no means a thoughtless repetition of a classmate's random wrong answer is shown by the following curious episode. Several days later the problem Serezha B. had (above) was given to this class:

> To decorate a fir tree the first-graders made 20 toys; six of them were paper and the others were cardboard. How many cardboard toys did they make?

When the teacher asked how to find how many toys were made of cardboard, one of the children answered, "Take 6 from 20." But all the other pupils in the class said "Aha!" in a friendly way and in unison, "Just the opposite." The correct (to us) method of solving the problem, given by the first boy, had seemed quite absurd to the rest of the class.

These observations, first, make it possible to maintain that the inability to choose the correct arithmetical operation or to write out the solution precisely is not necessarily connected with failure to understand the text of the problem. Second, they permit us to assume that the children have "their own" strictly defined methods of solving a problem, but these methods differ from those with which we adults solve these problems. Third, they force us to break down the very concept of "understanding." If the children thoroughly understand the situation described in a problem and the relationships between the parts of the object group and are still unable to select the proper arithmetical operation, there apparently are several different "understandings" of the situation in a problem and naturally several different "senses" in the problem itself. Some of them correspond to those methods the children use to solve the problem and others correspond to a socially fixed mathematical method, the kind that we adults have already mastered and with which we solve these problems.

Conclusions

These conclusions present us with three fundamental problems for investigation. We must ascertain:

1) What are these methods of solving arithmetic problems that the children use? Under what conditions and for solving what kinds of problems are they formulated?

2) What is our contemporary mathematical method for solving these problems? Under what conditions and for solving what kinds of problems is it formulated?

3) How should children be taught this socially fixed method for solving arithmetic problems?

Report II: Methods of Solution and the Content of Arithmetic Problems

Methods of Solution Used by Children Before They Have Mastered Addition and Subtraction

We know that in solving "indirect" arithmetic problems, many first- and second-graders make certain standard mistakes. When the sense of the problem indicates that they should add, they subtract, and when the mathematical sense of the problem demands subtraction, they will add. When we analyzed the processes of problem solving, (See Report I), we became convinced that it would be superficial to attempt to explain these mistakes by saying that the children do not understand the substance of such problems. Moreover, we noted rather frequent instances where children would give the correct answer to a problem almost immediately but would still write out the solution incorrectly and follow it with a second, incorrect answer, surrendering to the logic of what they had written. We thus concluded that children have "methods of their own" for solving such problems, different from our socially established method of solving them by means of addition and subtraction. We confront the task of analyzing the children's methods.

It is often rather difficult to disclose a method (or mechanism) of problem solving. Such responses as these are typical.

Kostya B., first grade, September

Experimenter: Ira had 8 stamps, some yellow and some blue. There were 4 yellow ones. How many blue stamps did Ira have?

Kostya: [whispers to himself] 8, 4. (A few seconds later): I know -- I just forgot: 4 and 4 is 8, so there were 4 blues ones, too.

Sasha B., first grade, September

Experimenter: There are 8 rabbits in two cages. In one cage there are 5 rabbits. How many are there in the other cage?

Sasha: Three.

E.: How did you find out?

P.: I thought a minute and knew.

E.: Did you work it out?

P.: No, I thought a minute and knew.

Clearly, such remarks are of no help in clarifying the actual mechanism of the activity. Therefore, we must find instances where a problem causes a child difficulty, where he is forced to externalize his method of solution in order to solve it. Sometimes, to disclose the method of solution, it works well to make use of supplementary accounts by the children.

We analyzed more than 40 cases where children verbalized their problem solving, and in them we found three categories or variants of solution methods which children use:

Variant A. The children reconstruct the groups of objects described in the problem (most frequently on their fingers, sometimes with blocks, counting sticks, or other objects), and then solve the problems with the aid of counting. These examples are typical:

Sasha Sh., first grade, September

Experimenter: There were some plums on a plate. A girl ate 6 of them, and then there were 3. How many plums had there been on the plate at the beginning?

Sasha: That's hard. I don't understand it.

E.: (Repeats the problem)

P.: (He holds up 3 fingers. Then, holding these three fingers along his nose, he holds up 6 more. He looks at them.) Nine.

Misha U., first grade, October

Experimenter: There were 7 dumplings. Some children ate some of them, and then there were 4. How many dumplings did the children eat?

Misha: (He had held up 7 fingers as soon as the experimenter began.) They ate 3.

E.: How did you find that out?

P.: I had 4 fingers together like this (he held up 4 fingers pressed together), and 3 like this (he hooks the thumb of one hand around the thumb and forefinger of the other hand).

Variant B. The children do not use anything to reconstruct the groups of objects described in the problem. They count the figures in a numerical sequence. Here are two examples:

Sasha B., first grade, September

Experimenter: There are 9 pencils in a box. Five are red, and the rest are green. How many green pencils are in the box?

Sasha: (Whispers something to himself; then after 41 seconds): Four.

E.: How did you find that?

P.: I counted.

E.: How?

P.: 6 - 1, 7 - 2, 8 - 3, and 9 - 4.

Vladik A., first grade, October

Experimenter: There were 7 glasses on a shelf. Then several of them got broken, and 2 were left. How many glasses got broken?

P.: (In 38 seconds) Five.

E.: How did you do it?

P.: I counted 1, 2, 3, 4, 5.

E.: How did you know when to stop? Do you think maybe you should go on?

P.: But 6 and 7 would be next.

(This example is somewhat different from the first, but we are assigning it to the same category. This will be discussed in greater detail in a later report.)

Variant C. As in the preceding category, the children move solely along the numerical sequence; yet they are not counting but are doing something which resembles addition and subtraction. Here is an example:

Zhenya G., first grade, December

Experimenter: A girl had 5 pencils; she was given several more and then she had 9. How many was she given?

Zhenya: Four.

E.: How did you work it out?

P.: I started with 5 and added 2 and then 2 more.

There were several cases of addition and subtraction by twos and one child added and subtracted by threes.

Having obtained several variants of children's methods of problem solving, we had to determine with which of them to begin the investigation. Only certain considerations concerning the genetic connections among these methods could be the basis for this. We assumed that variant A was genetically primary, and that variants B and C were subsequent transformations and developments of it. Furthermore, we assumed that the first method of behavior is closest to a simple count of collections of objects and therefore could be a natural and direct outgrowth of it. We thus faced the task of analyzing the method of solving arithmetic problems that is based on first reconstructing (or making a model of) the groups of objects described in the problem, and then counting.

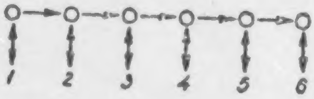
Counting and Transforming Objects in Groups: The Structure of the Problem

The theoretical basis of our analysis consisted of the concepts and principles of substantive-genetic logic [7,8], and specifically the idea of the organic connection between the two ways of regarding thought -- as cognition and as a process [5,8]. In this view, knowledge is considered as the substitution of signs for operations with objects [8: I; 9: I], and by virtue of this, as a two-dimensional structure that does not come under the principle of parallelism of form and content [9]. A logical analysis guided by these principles permitted us to examine behavior in problem solving as the "norm" or "method" of solution. This observation of behavior is a necessary premise for a psychological analysis of all children's learning activity [6] and, indeed, for a pedagogical analysis (in the narrow sense of the word) of the educator's teaching activity.

The initial component of problem solving is variant counting. This assumption served to qualify this method as genetically primary. The special problems involved in analyzing counting as a special mental activity and of the logical structure of the numerical sequence exceed the limits of this study. Here we wish to provide a cursory review of only those issues that are absolutely indispensable in the present context.

Counting is a socially elaborated and socially established method of solving certain problems on the level of objects. The problems themselves are expressed in questions or tasks of a particular sort, and they necessarily assume the existence of the objects themselves. We recognize this when we say that these problems are on the level of objects. There are only three types of problems here -- two partial and one integral.

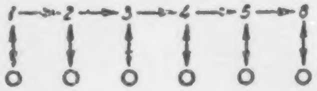
The first partial problem, "How many objects are there (on this table, in this room, etc.)?" always includes a precise indication of the spatial and temporal boundaries of the given field, the objects being immediately perceptible. The problem-solving process itself is a substitution, in a particular order, of figures for the objects in the group (or for the counting operations), a particular figure replacing each:



(1)

and a particular figure for the whole aggregate. In schematic form, this process can be represented: $X\Delta\uparrow^{(A)}$ where X is the aggregate of objects, (A) signifies the figures of the sequence, and $\Delta\uparrow$ -- the "delta-arrow" -- is the counting operation, including the series of comparisons [8: 44-45] and movements depicted in the foregoing scheme.

The second partial problem is: "Take or choose so many objects from the given aggregate." The solution process is again counting, but with a somewhat different connection between the objects and the figure. In the first problem the actual number of objects in the particular aggregate determined the figure one would get, but here the figure given at the start determines a chosen or created aggregate of objects. One might say that in a certain respect the operations employed in the first and second problems are inverse. The first one we shall call counting up the objects and the second, counting them out. Schematically the second operation can be depicted:

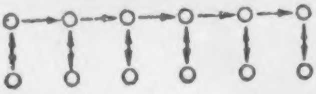


(2)

or $^{(A)}\downarrow\nabla$ Y, where (A) signifies the figures of the sequence, Y is the aggregate being counted out or reconstructed, and $\downarrow\nabla$ (the "inverted arrow") is the operation of counting out.

The integral problem is: "Set aside or choose from among the objects of the given aggregate the same number as there are in the other aggregate." The solution of this problem assumes both counting operations -- counting up and counting out. The whole process can be depicted by a combination of schemes (1) and (2), or by the formula $X \Delta \uparrow^{(A)} \downarrow \nabla Y$.

Let us emphasize that from the point of view of logical origins, it is the latter, integral problem that is the original one. It occurs purely on the level of objects and is formulated approximately this way: "Set up an aggregate of objects Y equivalent to the aggregate of objects X ." Originally it is solved not by counting, but in essense purely with objects. This operation might be depicted schematically in this way:



(3).

In diagram form the solution of such problems can also be depicted as X ⟶ Y. Only in certain condition, in so-called "rupture situations," when the problem cannot be solved by this method, it begins to be solved in another, indirect way, using substitutes (objects or symbols). Counting makes its appearance as a separate activity in precisely these situations, and process X ⟶ Y is transformed into process $X \Delta \uparrow^{(A)} \downarrow \nabla Y$. But even when the structure has been complicated in this way, the process of solving the primary problem -- of "setting up an aggregate of objects Y equivalent to the aggregate of objects X" -- remains originally an integral unit, a single operation, one might even say, and only subsequently is it separated into two operations that are relatively self-sufficient and, seemingly, largely independent of each other. The product of the first operation is a definite number, which originally had no practical meaning of itself, was only an intermediary means for solving a practical problem with objects, and thus appeared insignificant and hardly necessary. But now, when the operations are separated, this number is transformed into a thing of independent value; it becomes the result that is sought after for its own sake.

This change in the significance of the symbol -- its transformation from an intermediary means into a special product -- at the same time isolates (and highlights) new problems which become just as important as the original practical ones. "Determine the number of objects that are here" and "Set apart the number of objects indicated by this number" are the wordings of these new problems, and they differ substantially, although at first glance only slightly, from the wording of the original problems. Isolating such problems completes the process of separating *cognitive operations* from *practical* ones (in this area). The former yield as their product certain *knowledge*, i.e., $X\ \Delta\uparrow^{(A)}$, while the latter yield a certain *aggregate of objects assembled on the basis of that knowledge*, $^{(A)}\downarrow\nabla\ Y$. In the case under consideration, the cognitive operation is counting up, and the practical one is counting out.

This whole process is also very closely linked with a division of labor, i.e., the distribution of the various parts of the original operation to different persons. One person counts up the given aggregate of objects, and another, when he has learned the results of the first person's activity -- a number -- counts out an "equal" aggregate. One might say that only when the activity is apportioned to different persons in this way are intermediate results isolated and the separate tasks of obtaining these results distinguished.

Counting, as a special activity directed toward solving the problems described above, "applies" to another kind of transformation of aggregates of objects -- combining and dividing them. It accommodates itself to this activity and begins to "work" in its context.

These two transformations of objects -- dividing and combining them -- can be depicted thus:

$$\begin{tikzcd} X \arrow[rr, bend right] & y \arrow[r] & Z \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} y \arrow[rr, bend right] & Z \arrow[r] & X \end{tikzcd} \qquad (4)$$

They structure reality in a definite way, creating two situations sharply separated from each other in time. While one situation is in existence, before the beginning of the transformation, let us say, the other one cannot exist, and when the second situation has come into being, after the transformation, then the first one can no longer be. For example, say we are dividing aggregate X into two parts; when there is a whole there will be no parts, and when the parts are formed there will no longer be a whole. It would be the same if two aggregates were joined. Diagrammatically the relationships formed here can be depicted this way:

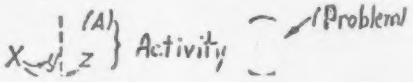
$$\begin{tikzcd}[column sep=small]
y \arrow[rrr, bend right] & z \arrow[rr] & \vdots & x, & & x \arrow[rr] \arrow[rrr, bend right] & \vdots & y & z & & x \arrow[r] \arrow[rrr, bend right] & y & \vdots & z & (5)
\end{tikzcd}$$

(The vertical dotted line in all these formulas represents the spatial-temporal boundary of the situations. The last formula corresponds to the case when the original whole is divided into parts, but one part disappears and only the other part is actually involved in the second situation.)

In actuality, however, there is a whole series of problems requiring a definite comparison of the results of the second situation with those of the first. For instance, in the first variant of the transformations (5) such a necessity could arise in connection with the question of what part of the whole X was contributed by participants A and B or in connection with whether the general quantitative character of the aggregate changed when y and z were joined to it. A similar question could arise with the second variant, too, but now concerning the division of X into parts, and so on. In all of these cases the first and second situations must be compared in order to answer the questions.

But such a comparison is possible only when something remains from the first situation and is carried over into the second. In principle the impossible must happen: The whole first situation must be preserved and transferred to the second. This is impossible, for if the first situation exists, then the second cannot, and vice versa. The solution is to introduce substitutes (objects or symbols). The first situation cannot be preserved; it disappears when it is transformed into the second. Substitutes or representatives need to be retained and carried over into the second; they need to be such that the necessary comparison of situations can be made. This, it is important to remark, is exactly what defines the relationship in a situation between objects and their substitutes. The substitutes are such only relative to the problem, and they reflect, take upon themselves, or convey only those properties of the objects that are necessary for the particular comparison called for by the problem.

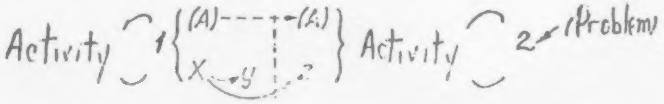
Depending on what the question is and which of the possible substitutes for the first situation we have, different problems can result from the same transformation of objects. The substitutes for the first situation and the elements of the second one form the conditions of the problem in a given instance. Thus the conditions of a practical problem dealing with objects consist of those objects of the second situation and of the substitutes for the first situation which permit a comparison, so that the problem can be solved. Comparing the objects of the second situation with the symbolic substitutes for the first is a special activity, and not such a simple one at that, for it is impossible to compare a number and an aggregate of objects directly. Thus, this activity obviously depends somewhat on the problem. Diagrammatically it can be depicted this way:



(6)

(Here (A) is the number determining the quantity of elements in aggregate X, and the bracket before the word "activity" indicates that a comparison is being carried out.)

But the substitutes being carried over from the first situation into the second had to be obtained there first. And this, too, was a definite activity of a special sort, intended from the outset precisely for creating substitutes which could be carried over into the second situation. If we take this feature into account, our formula will look like this:



(7)

It is important to note especially that "activity 2," by means of which the objects and symbols of the second situation are compared, depends on three features: 1) the character of the transformation of the objects in the aggregate, 2) the problem, which is determined by the broader, real-life situation, and 3) the nature of the substitutes obtained from the first situation and transferred into the second. "Activity 1," by means of which the substitutes are obtained in the first situation, in turn also depend on three features: 1) the character of the transformation of the objects in the aggregate, 2) the possible character of "activity 2," and thus indirectly the problem, too, and 3) certain incidental circumstances determined by the broader, real-life situation. For instance, if it was impossible to devise a substitute for all of aggregate X, an obtainable substitute for part of y could make up for it, or the like.

In this system of relationships it is especially important for us to emphasize: 1) the dependence of "activity 1" on "activity 2" or the dependent of what is done first on what will follow, and 2) the mediating role of the part of the conditions presented in symbolic substitutes. These problems serve to connect activities 1 and 2 into a single integrated activity for solving the particular practical problem, and consequently they must be set up so as to provide this connection. In other words, this part of the conditions of the problem fulfills a certain function in the activity, and it should be tailored to this function.

If the conditions of the problem can assure a connection between activities 1 and 2, then in principle it becomes possible to divide up or distribute these activities to different persons:
One person, then, can be creating substitutes for the first situation, while another person, in another time and place, is doing nothing but comparing these with the aggregate of objects in the second situation and solving the problem. This becomes completely feasible if we further supplement the conditions of the problem by including a description of the transformation of the objects in the aggregates. This will permit the second person to reconstruct

the object part of the first situation, to relate correctly his aggregate of objects to the ones in the first situation, and on this basis to choose the correct type of comparison between the aggregates of objects he has and the symbolic substitutes for the others. If the practical activity is divided between different people without such a supplement, the problem cannot be solved, as the second person, not having directly observed the transformation of objects in the aggregate, cannot even qualify the aggregate assigned to him. It could just as easily be a part as the entire whole. Supplementing the conditions of the problem by describing the transformation of objects in the aggregates brings the problem closer to its textbook form, with which we are usually concerned (although even this approximation is not complete, since the object element z is still present).

Report III: Variants in Solving Problems Presented with Objects

Introduction

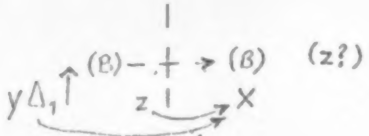
When we analyzed the causes of first graders' difficulties in solving simple arithmetic problems, we discovered that, in addition to the customary method of solving them by adding and subtracting numbers, they use at least three other methods of solving these problems: a) by using objects to make a model of the aggregates described in the problem and then counting them up, b) by counting up the figures in the number sequence, and c) by "adding" or "taking away" figures in the number sequence by twos or threes. The first method of solution -- by making a model with objects and counting them -- we singled out as being genetically primary; from it (in the conditions of existing instruction) children move on to the second and third methods or directly to the socially accepted, standard method. But this genetically primary method of solution is itself sufficiently complex. The children arrive at it gradually, too, through even simpler modes of activity. Much knowledge and many mental operations are wrapped up in it, too, and therefore it is not easy to analyze its structure. In order to overcome these difficulties and to analyze the structure of the solution processes, we introduced a special (in a certain sense fictitious) model of an arithmetic problem -- "problem presented with objects." In its design, this is a problem which could crop up directly in the context of practical activity, when actual aggregates are being broken down and combined, and it presupposes the actual presence of certain parts of these aggregates; the latter, as it were, enter into the conditions of the problem itself along with the symbols. When we analyzed these genetically simplified models, we were able to isolate a number of important aspects of the present-day textbook arithmetic problem and examine them apart from other secondary aspects. Three factors in particular emerged in especially sharp relief as ones on which problem solving activity depends: a) the character of the transformation of objects in the aggregate, b) what the problem is asking, and c) the character of the substitutes (symbols) found in its conditions (Report II).

But it turned out at the same time that these models, introduced originally, we repeat, as a kind of abstract, fictitious prefiguring of real arithmetic problems, correspond to completely real problems which are (or in teaching can be made) genetically primary arithmetic problems. We tested this thesis in experiments with preschoolers and obtained a number of important results, to be presented elsewhere. But here, having simply noted this fact of subsequent experimental verification, we need to set forth the basic features of the theoretical analysis of the possible methods of solving problems presented with objects. As we do this, we want to pay special attention to the method we use of diagramming the processes of problem solving (Report II). As a matter of fact, for us the diagrams emerge as abstract models of the actual solution processes. When we analyze them we get all kinds of information about the peculiarities of children's problem solving. Without consulting the empirical material directly, we anticipate the results of the experiment. This information, obtained in the diagram-models, was subsequently confirmed by the experiments with preschoolers.

Variants for Problems Presented with Objects

What first becomes clear from the scheme of the problems presented with objects is that the solution of each of its variants can proceed on two planes -- of objects or numbers -- and correspondingly the solution processes will differ substantially in both the operations involved and the "understanding" of the conditions which these operations determine.

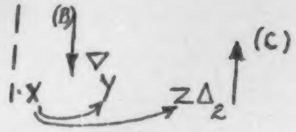
By way of illustration, let us take the first type of problem, when the two aggregates y and z have been combined into one; we have the combined aggregate X directly before us here, we know the number characterizing the quantity of elements in one of its parts, and we must either take away the second part physically, or express the quantity of elements in it by a number. This type of problem can be expressed schematically in Formula 1:



Formula 1

where the vertical dotted line represents the temporal division of the situations, $\Delta_1\uparrow$ (the "delta one-arrow") is the operation of counting up, and (z?) is what the problem is asking (Report II).

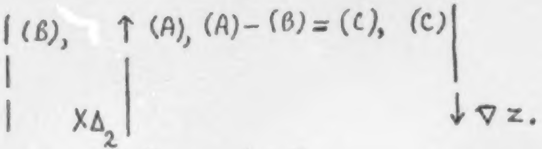
If we are going to use objects to solve the problem, then from the aggregate X directly before us we will have to count out the aggregate corresponding to the number (B) , i.e., aggregate y, at the same time take away aggregate z from X and, if the question calls for it, count it up and obtain number (C). This solution



Formula 2

process can be represented by Formula 2. The symbol for counting out (read "inverted arrow") in this formula, taken together with the symbol for dividing aggregate X, signifies the removal of part y from X.

But if we are going to be using mainly numbers in solving this problem, then we will have to count up the aggregate X immediately before us, subtract number (B) from the number (A) thus obtained, and then count out aggregate z if the problem calls for it. This solution method can be represented by Formula 3.

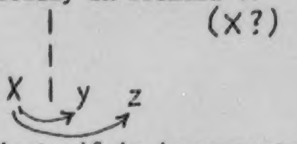


Formula 3

When we compare these two methods (Formula 2 and Formula 3) of solving the same problem (presented as in Formula 1, we emphasize, with objects), we can readily see that the first method (Formula 2) based on moving the objects themselves, is unquestionably easier, more natural, and economical than the second. It contains just the one operation of counting out, if we want to obtain the aggregate of objects z, and two operations -- counting out and counting up -- if we want to obtain the number characterizing aggregate z. The second method (Formula 3) contains either three operations -- counting up, subtracting, and counting out, or two -- counting up and subtracting. It should be added that, in what it comprises, the operation of counting up in the second case is equal to both the counting out and counting up operations in the first case.

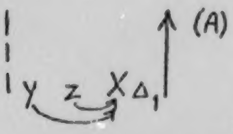
It is quite obvious that from the standpoint of the logic of of problem solving, the second type of problem, where the numerical value of aggregate z is known and that of aggregate y unknown, coincides completely with the preceding variant. This similarity of the problem types is substantially what distinguishes problems presented with objects from purely arithmetical, textbook problems.

Let us examine the third type of problem, when we have both of the partial aggregates directly before us, and we must either form a combined aggregate or determine its numerical value. It is represented schematically in Formula 4:

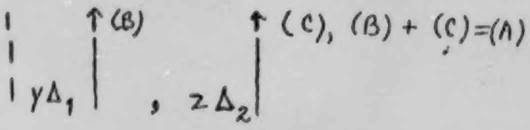


Formula 4

In essence this variant, if it is presented with objects, does not yield a really arithmetical problem at all. There are two modes of activity possible here: 1) We can combine aggregates y and z (either actually or in a representation, on an "understood" level) and count up the aggregate obtained; or 2) we can count up the aggregates presented separately and then add the numbers obtained. These two modes of activity can be represented by Formulas 5 and 6:



Formula 5

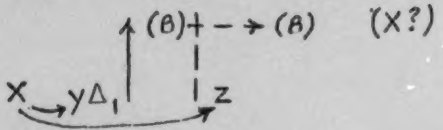


Formula 6

It is readily evident that here, too, just as in the first and second types of problems, the solution using the objects themselves turns out to be easier and more economical than the one based on the manipulation of numbers. It will suffice to point out that the operations of counting up aggregates y and z are equal, in effort required, to the operation of counting up all of aggregate X, whereas in the second instance addition, too, is required.

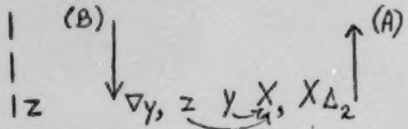
Let us turn now to the following types of problems, The aggregate has been divided and we have only one part immediately before us. Two situations are possible: 1) We know the numerical value of the second part and must determine the whole; or 2) We know the numerical value of the whole and must determine a part. In essence these are two completely different tasks, and solving them requires different activities. We might call these the fourth and fifth types of problems. Let us examine them in order.

The fourth type of problem can be represented by Formula 7:



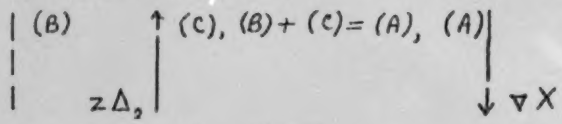
Formula 7

If we wish to use objects to solve the problem, we must first introduce, as a supplement to the conditions, an auxiliary aggregate of objects (sticks, fingers, etc.) from which we will take objects to reconstruct the missing parts of the original aggregate described in the conditions of the problem.[1] Then the solution of problems of this type will proceed thus: First we will count out aggregate y, then physically combine y and z into one aggregate and finally count it up. The solution process can be represented by Formula 8.



Formula 8

But if it is going to be solved mainly with numbers, then we will first have to count up aggregate z, then add the number we get to the one we had, and finally, if the question calls for it, count out the combined aggregate, Diagrammatically this process is represented by Formula 9:



Formula 9

This is the only type of problem where both methods of solution -- with objects and with numbers -- are approximately equivalent from a general standpoint. The first method gains the advantage when the solution to the problem is the formation of an aggregate of objects, but the second does if the answer is to be given in the form of a number. In specific cases the superiority of one or the other also depends on the correlation between the quantities of objects in the aggregates y and z.

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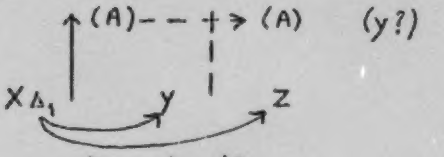
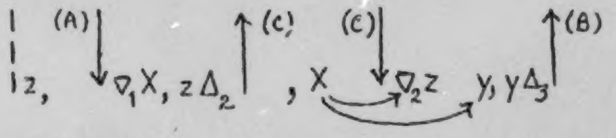
[1] Let us emphasize that we are not doing to examine the modeling operations themselves here. Subsequent reports will be devoted to a more detailed analysis of these. Thus -- and it is important to keep this in mind -- depicted this way, the problem-solving process is still being considered without its full complement of mental operations.

The fifth type of problem is represented by Formula 10.



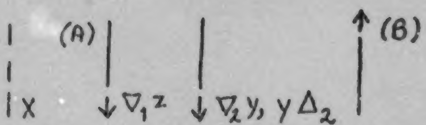
Formula 10

It is the most complex type: At least two substantially different methods of solving it with objects are possible. In the one instance, using auxiliary objects, we first need to count out an aggregate of objects X in accordance with number (A), then count up aggregate z, which is given in the conditions, and having obtained the number characterizing it, count out an equal aggregate within aggregate X; in the same way we will set apart aggregate y within X and then count it up. Diagrammatically this very intricate process can be represented by Formula 11.



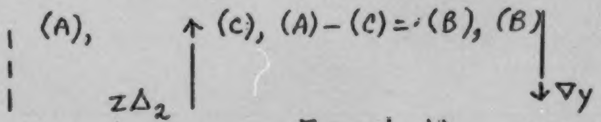
Formula 11

A method of solving the same problem, which is simpler in terms of the number of operations but at the same time more "profound" (from the point of view of "understanding" and the mechanisms of activity involved in it), consists of counting out aggregate X; then a continuation of the counting out, beyond the limits of aggregate z, i.e., with auxiliary objects, will yield an aggregate of objects y, which can be counted up afterward. This problem-solving process can be represented by Formula 12.



Formula 12

The third method of solving this problem, with numbers, will consist of counting up aggregate z, subtracting the number obtained form the given number (A), and, if the question calls for it, counting out aggregate y. This process can be represented by Formula 13.



Formula 13

It is readily evident that this is the only form of the problem which is more complicated to solve with objects (the first way) than with numbers. The second way of solving it with objects turns out, from the standpoint of the number of operations, to be simpler than the method using numbers, but it assumes a very high level of "understanding" of the relations between aggregates of objects (we shall discuss this in a subsequent report) and therefore will undoubtedly prove difficult for children.

Summary

As we complete this analysis of possible methods of solving arithmetic problems presented with objects, we want to emphasize one feature in particular. When we compared the problem-solving methods using objects and numbers, we always worked from the assumption that the person performing the activity actually had the objects necessary for making a model of the aggregates. This assumption is completely justified when we are analyzing abstract models of problems done at school. For at school, children at the first stages very often make use of object models — counting sticks, things used as abstract objects, etc. We found it important to ascertain that the methods of problem solving using objects turn out to be more advantageous in these conditions than solving problems through the use of numbers. But if we abandon this initial premise, if we assume that the person has no auxiliary objects but only the matter of the initial aggregates that are being transformed, then it will turn out that only three problems -- of the first, second, and third types -- can be solved with objects at all, and the other two -- the fourth and fifth -- absolutely require a solution using numbers.

This observation has very important implications for teaching: It specifies more precisely the conditions necessary for organizing children's mastery of the modes of activity described above. In particular, it isolates the problems that can place children in a rupture situation.*

Report IV: Problem Solving by Making a Model with Objects and Counting: General Characteristics of the Method and the Basic Problems Which Arise in Investigating It

Introduction

In the preceding report we examined models of arithmetic problems presented with objects: Parts of the actual aggregates of objects to be transformed entered into the conditions of these problems along with the numbers. The presentation of these problems with objects made it possible to use counting in solving them and to transform the objects in the aggregates. Contemporary arithmetic problems are totally different from those presented with objects. They are completely removed from the plane of objects; their conditions include only numbers (at least two) and descriptions of the transformations undergone by the aggregates of objects. These changes in the conditions entail a change in the activity by means of which the problems are solved, as well. In the problems presented with objects, it was possible to count up the aggregates, combine them (or separate them), and count them up again while determining the numerical value of the aggregates thus created or destroyed. In a textbook arithmetic problem it is unnecessary -- impossible in fact -- to count anything up. Everything needed for the solution is already counted up, and there are no objects as such at all. The mode of activity adequate for this problem consists of the formal mathematical operations of addition and subtraction. Man worked these out at a certain stage in his historical development and has since handed them down from generation to generation. Learning to solve arithmetic problems means mastering the method of solving them by addition and subtraction. This method itself is a complicated matter and consists of more than the formal operations of addition and subtraction (as will be shown in subsequent reports; see also footnote 2 on the next page).

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*A rupture situation is one where a child is confronted with problems for which he has not yet learned appropriate modes of activity to solve the problems (Ed.).

Furthermore, mastering addition and subtraction is complex, too, with its own particular laws. We can scarcely discuss these laws at present with confidence. We do not even know whether the learning involves the transformation of modes of activity the child already has into a new mode, or the "pure" acquisition of a new mode brought in, as it were, from without, largely irrespective of the modes of activity he already knows. But every time a child is confronted with a problem requiring a new method of solution, he tries to solve it first through the methods he already knows. Thus, independent of what the "pure" mechanisms of actual mastery are, the new problem is always "refracted" through the prism of available methods of solution, and we should take this into account in our investigations.

This applies fully to the processes of solving arithmetic problems as well. When children are given a strictly arithmetical problem for the first time, in essence they are put into a rupture situation: Solving the problem calls for a new mode of activity which the children do not yet have.[2] In this situation, with the teacher urging them to solve the problem, try to make use of the modes of activity they already know -- counting, in particular -- and adapt them to the new conditions. But to do this they must turn from the arithmetic problem presented with numbers to the problem presented with objects and supplement it with aggregates of objects. So the children introduce auxiliary objects (their fingers, for instance)[3] and use them to reconstruct the aggregates of objects corresponding to the numbers given in the problem, thus making models of the original aggregates of objects and their transformations.

But in doing this they do not simply use a mode of activity that they have already mastered -- counting, for example -- but rather they work out[4] what in fact is a new mode of activity, a combination of previous ones, with the initial elements of the activity somewhat modified and transformed. Quite typical is Sasha Sh.'s behavior discussed in Report II, when he is given the problem: "There were some plums on a plate. A girl ate 6 of them, and then there were 3. How many plums were there on the plate to begin with?" First, he says the problem is "hard. I don't understand it," and then he solves it, by holding up first three fingers, then next to them, six more, and finally counting them all up. His difficulty, obviously, was not in reconstructing an aggregate of objects according to the specified numbers, but in reconstructing the aggregates in the relationships that correspond to the conditions of the problem. The fact of the matter is that making a model of the situation described in the conditions includes two consecutively performed counting-out operations, and even in elementary cases, when the first aggregate of objects has been reconstructed according to one of the numbers, it is then necessary to determine how or where the aggregate corresponding to the second number is to be reconstructed. Let us illustrate this with a very simple example. A problem is posed: "There were 7 birds in a tree . . ."; immediately the child holds up seven fingers, but then, depending on what happened in the situation described -- whether more birds came or some flew away -- he is going to have to count out a second quantity either side by side with the first, continuing and supplementing it, or in the "opposite" direction, "within" the first aggregate. This choice, depending on the nature of the problem and assuming a certain "understanding" of its conditions, is precisely that new feature which distinguishes this activity fron the simple counting of objects that was mastered earlier, and this is just what children initially have difficulty grasping. (All of this is only a superficial description; a more detailed and precise analysis of all the points mentioned here will be given gradually in the course of further analysis.)

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[2] So that children will not be in a rupture situation when they are first given arithmetic problems, and so that they will not "invent" their own methods of solution, they are being taught the operations of addition and subtraction often even before they are given the first arithmetic problems. This is instruction in solving so-called "examples." But our observations in Report I show that children who can solve the arithmetical examples well are still unable to solve many problems. This permits us to conclude that how we solve arithmetic problems is more than addition and subtraction. Even those children who have mastered these formal operations get into a rupture situation when they confront problems.

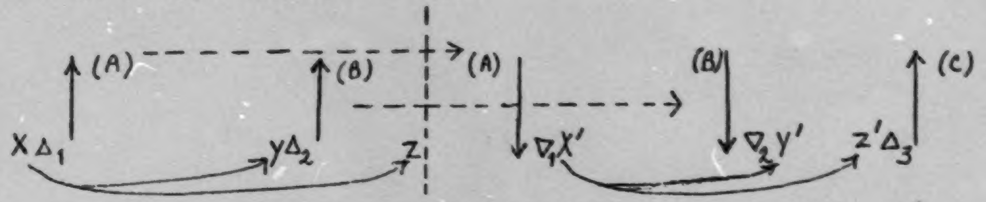
This conclusion determines the problems to be investigated further. How are the solution of examples and the solution of problems connected? What else, besides addition and subtraction proper, enters into the method of solving problems? When we have answered these questions, we will then be able to ask whether instruction in solving examples might not be organized to provide simultaneously for mastering everything needed for solving problems. Clearly, these questions should be answered by analyzing the method itself, based on addition and subtraction, but analyzing the genetically simpler methods of making models with objects will shed some light on them, too.

[3] The conditions in which they do this will be discussed in subsequent reports.

[4] See the more precise description of this in the final section of this report.

Solution by Making a Model

A major circumstance, specifically, is that this method of problem solving is based on a special substitution -- making a model, in the precise and narrow sense of the word. If we diagram the child's activity when he is solving any elementary problem, it will appear approximately this way:



(X, y, and z designate here the aggregates of objects, the symbol $\Delta\uparrow$ -- the "delta arrow" -- signifies counting them up, the symbol $\downarrow\nabla$ -- the "inverted arrow" -- signifies counting out the aggregates by number, and the curved arrows represent the decomposition of aggregate X into parts y and z. More detailed explanations of these symbols were given in Reports II and III.) In the first situation the numerical value of aggregate z was not given. This constitutes the question being asked. To answer it, the child must reconstruct, in accordance with the numerical value of (A), whe whole aggregate X which was divided in the first situation, but now with other objects, i.e., aggregate X', and then within it reconstruct with the new objects the partial aggregate y' according to number (B), thus essentially repeating in the second situation and with new objects the same division of the aggregate which took place in the first situation. The aggregate z' obtained as a "remainder" will correspond to the original aggregate z, and therefore, when he has counted up z', he will be able to transfer the number obtained to the original aggregate z. The question is answered, though not as a result of counting up the original aggregate z to which the question actually pertains, but as a result of counting up another aggregate, z'. But this other aggregate is such (actually, it is so set up) that the results obtained with it can be transferred to the original aggregate. Another important feature here is that precisely the same operation of counting up is applicable to the newly created aggregate z' as was to be applied to the original aggregate z. These two features: 1) the application to z' of the same operation as was to be applied to z, and 2) the transfer to z of the results obtained from operating with z', form the specific character of *the model as a special type of substitute*. Precisely by virtue of these two circumstances, z' is a *model* with respect to z, while z is a *pattern* with respect to z'.

If we extend this definition from the result to the whole activity by means of which the result is obtained, we can say that this whole activity is a *modeling* of the original aggregates of objects described in the problem and of their transformation. But at the same time it must be remembered that this definition has as its basis a comparison of only the final operations of this whole activity, which as a whole, is model-making insofar as it is directed toward obtaining a model of what is asked about in the problem. It would however, be incorrect to look for a model-pattern relationship in all the elements and components of this activity. In particular, it would be wrong to attempt to interpret the successive operations of reconstructing the aggregates of objects according to the numbers in the second situation as a modeling of the object transformations that took place in the first situation, as the problem variant we cited above suggests. Later on we will see that, depending on the form of the problem itself, the relationships between the operations of modeling aggregates of objects and the object transformations in these aggregates are quite complicated and variable. Nevertheless -- and this, too, will be shown later on -- children unwittingly, but very frequently, perceive precisely such relationships and begin to pattern their activity on them. Therefore, it becomes quite important to try to prevent such a misinterpretation.

Learning the Method of Making a Model

We also need to discuss the degree to which children *learn* the above-mentioned method of modeling the conditions of the arithmetic problem, and the degree to which the "invent," "discover," or construct it.

Just as with any other mode of activity, the solution of arithmetic problems through making a model with objects has as its basis the learning of definite modes of activity worked out by mankind and "imparted" to the child in a special way.

These assertions apparently are indisputable with regard to counting. But do they extend also the the "addition" specific to such a solution of an arithmetic problem -- the modeling with objects and the determination of the order of reconstruction operations? After all, as special operations, counting up and counting out take shape in connection with the solution of somewhat different problems related to the object level proper. Likewise, children master them through other problems. In order to use these methods of operation here, in textbook arithmetic problems, the child must transform them radically. Not only that, but the very "idea" of using objects to make a model of the conditions is a substantial addition that apparently must also be "discovered" or else learned in specially organized instruction. Special research is needed to give a well-substantiated answer to these questions. In particular, it is necessary to clarify in detail the way counting is taught, and whether situations are not already created that lead to essentially the same problems, but on the level of objects. Are not the elements and the general scheme of making models with objects worked out even before we come to arithemtic problems proper - for instance, in problems presented with objects, or even in ordinary counting? If this is found to be so, then of course we cannot speak of the child's "discovery" of modes of activity here either, but will have to talk about direct mastery.

But right now we are interested even more in another side of the matter. In principle we apparently cannot and should not deny the possibility of the child's constructing solutions to problems. Moreover, this is what we should strive for, developing in children the ability to construct solution processes on their own and then to turn such processes into solution methods. The actual problem, therefore, consists of finding the limits of this activity of the child's on his own and in seeing how his construction of solution processes is related both to the modes of activity he has already learned and to new modes he has found on the basis of the constructed solution. In essence we shall be concerned with this range of questions throughout all our work, but in addition it will be the topic of a special discussion in one of the subsequent reports.

Report V: *Solution by Making a Model with Objects and Counting*: *A Theoretical Analysis of the Problem Variants*

Introduction

In the previous report we examined the relationship between *textbook arithmetic problems* and so-called *problems* "*presented with objects*" and gave a general description of the method of solving textbook problems by *making a model with objects and counting*. As we showed, the heart of this solution method is the use of some other kind of auxiliary aggregates to *make a model* of the situation described in the problem. And the condition for making a model is -- to use traditional terminology for the time being -- a certain *understanding* of the text of the problem. Only on the basis of this understanding can the child choose *the direction in which to count out* the second aggregate.

An analysis of the experimental material from this standpoint reveals what at first glance seems strange: The same children who fully understand problems of some types (accordingly, they know how to solve them) do not understand problems of other types at all.

Here is a pertinent set of observations:

Sveta M., *first grade*, *October*

Experimenter: A boy had 7 pencils. He lost 2. How many did he have left?

Sveta: (immediately) Five.

E.: There were two white goslings and some yellow ones in the courtyard. There were four goslings altogether. How many yellow ones were there?

P.: (thinks for a long time) Six.

E.: A cat had some black kittens and two gray ones. Altogether there were five. How many black ones were there?

P.: (counts on her fingers) Seven.

Lyuba L., first grade, December

Experimenter: Grandma made dumplings. Vera ate two.

(Lyuba holds up two fingers of one hand.) And there were five left for Mama ... (Lyuba holds up all the fingers of the other hand.) ... How many dumplings did Grandma make?

Lyuba: (counts up her fingers) Seven.

E.: First there were some birds, and the four more came (Lyuba holds up four fingers.) And then there were seven. How many birds were there to begin with?

S.: First there were four and then there were seven altogether. Seven birds, right?

E.: (Repeats the problem.)

S.: (again she holds up four fingers) What do you mean? I don't understand. There were four, but seven didn't come.

E.: (Repeats the problem for the third time. Again Lyuba does not solve the problem.)

E.: First there were some books, and somebody brought two more, and then there were five. How many books were there to begin with?

S.: (holds up two fingers on one hand, then all the fingers of the other, and then counts them all up) Seven.

It appears as though the children being tested understand the first problem and do not understand the second or third. But what is the difference between these problems? Why do these two girls (and many other children, whose records we have not cited) understand problems of one type fully and not understand problems of another type at all? What is the essential difference between these problems that makes for such a strange disparity in children's reaction to them? And what actually is this understanding?

While the child is hearing an arithmetic problem being read, such as, "There were some birds in a tree. Six more came, and then there were 11...," he can imagine a real tree with birds fluttering around in the branches (or a picture of a tree with birds sitting in the branches, of the type frequently provided of late in textbooks). Then he will imagine birds flying towards the tree and alighting on its branches, finally, in accordance with the text, the tree with the birds settling down after their flight. This entire process of imagining different situations in succession is undoubtedly a definite understanding of the text and the events described there. But it this the kind of understanding needed for solving arithmetic problems? Understanding the problem is, after all, only one step in the solution process. Certain operations -- problem solving proper -- must be carried out on the basis of it. In the cases we are discussing this will apparently consist of making a model, out of certain aggregates of objects, of the situation described in the problem. This activity presupposes understanding. Not only that, but it is apparently possible to say that this understanding is itself achieved through the model-making activity. It is needed only in order that the solution be obtained with the model, and it should be such that this function it has is guaranteed. We can present this schematically:

Understanding of the conditions of the problem ← model-making

But one might ask whether the understanding-imagining described above is that understanding which guarantees subsequent model-making ability, and if not, then what must it be? In order to answer these questions, we must analyze the structure of the model-making activity needed to solve various arithmetic problems.

Variants of Solution by Making a Model

The process of making models of simple arithmetic problems with objects has its own strict logic that depends on which of the aggregates in the conditions of the problem are known and which are not. If we consider all the problems from the point of view of *the character of the transformations of the objects described in the conditions and the sequence* in which the known and unknown quantities are presented, there will be seven variants in all. If we use the symbol γ ("the symbol of union") to represent the combining of aggregates as described in the conditions, the symbol $\wedge$ ("the symbol of intersection") to represent the division or isolation of aggregates as described in the conditions, the symbol (A) to represent known whole quantities, the symbols (B) and (C) to represent known quantities of parts, and the symbol (?) for unknowns, then these seven variants of the conditions can be diagrammed this way:

$$\begin{array}{llll} 1)\ (B)\ \gamma\ (C) \longrightarrow (?) & 3)\ (A) \wedge (?) \longrightarrow (C) & 5)\ (B)\ \gamma\ (?) \longrightarrow (A) & \multirow{2}{*}{$7)\ (A) < \begin{matrix}(?)\\(C)\end{matrix}$} \\ 2)\ (A) \wedge (B) \longrightarrow (?) & 4)\ (?)\ \gamma\ (C) \longrightarrow (A) & 6)\ (?) \wedge (B) \longrightarrow (C) & \end{array}.$$

The seventh variant can conditionally be called neutral. It does not show how the aggregates are transformed in it. It simply states that there are so many objects altogether and some are of one sort and the others are of another sort.

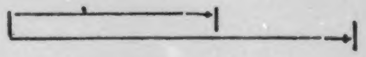
Let us now examine these problem variants from the point of view of the possibility of solving them by making a model with objects and counting. As we do this we shall be paying particular attention to two matters: 1) the relationship between the sequence in which the known and unknown quantities are presented in the conditions, on the one hand, and the possible sequence in which models of the aggregates of objects can be made, on the other; and 2) the character of the transformation of the aggregates described, on the one hand, and the character of the transformation of the models, on the other.

In the first and second problem variants, the sequence in which the known values are presented coincides completely with the sequence in which the models are made with aggregates of objects. As the child hears the conditions of the problem, he can immediately form an aggregate of objects corresponding to the first number. Then he must determine the direction in which to count out the second aggregate. The words "flew away," "came," "altogether," "of them," etc. can serve as points of reference here (we are leaving aside now the question of whether this way of solving the problems is justified and acceptable from a broader point of view. What is important is that with these variants the children are able to operate thus). When he has formed the second aggregate, the child automatically gets a third one -- the whole or a part -- which he can count up. These problems are obviously the simplest ones, and an analysis of the difficulties that they can cause children should either be done with the weakest children or brought down considerably to the preschool level.

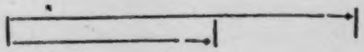
The third variant evidently should not cause particular difficulty either. Here, too, the child begins by forming an aggregate of objects corresponding to the first number, then simply skips the unknown and, guided by the same words -- "flew away," "altogether," "divided," etc.,--forms an aggregate corresponding to the second number, obtaining as a remainder the aggregate corresponding to the unknown number. Thus the third variants should be solved in the same way as the second.

The sixth variant, if we take it from the point of view of the sequence in which models of the quantities are made, should not cause difficulty either. As the child hears or reads the problem, he will skip the first unknown, then in succession form the aggregates corresponding to the first and second numbers, and as a result obtain the unknown number. But if it is easy from the standpoint of the sequence in which the aggregates are formed, this variant should present a certain difficulty from the standpoint of choosing the direction in which to count out the second aggregate. Here the words "flew away," "ate up," "altogether," etc. can no longer be points of reference. The child must perform a certain transformation in the conditions of the problem. He must begin to proceed in reverse order as it were. When he has formed the first aggregate he must then ask himself how the second one should relate to it. This transformation or, in other words, the answer to such a question, should obviously comprise the understanding of this problem variant.

But in the fourth problem variant this aspect of the matter emerges with particular clarity. The child skips the first mention of an aggregate, forms an aggregate corresponding to the first number, and then finds himself confronted with a terrible difficulty. He does not know what to do with the second number, *how* and *where* to form an aggregate corresponding to it. Making a model, in the sequence in which the known numerical values are presented, presupposes an exceptionally deep (and indirect) understanding of the relationships between the corresponding aggregates. Inasmuch as the aggregate just counted out, corresponding to number (C), is a part of the second aggregate, corresponding to number (A), the child should have begun to count it out a second time. Diagrammatically it would look this way:



Another way is considerably more natural: that of turning the conditions of the problem around. Here one would begin by forming an aggregate corresponding to the second of the numbers given -- (A), and then within it count out an aggregate corresponding to the first number (C). This sequence of operations can be diagrammed thus:



But this procedure as well presupposes quite a special "understanding" of the conditions of the problem. *Even before beginning the actual operations of model-making and counting out*, one must determine the relationship between the aggregates corresponding to the first and second numbers. This is the relationship between whole and part, and in order for the child to understand problems of this type, he needs to have already formed a concept of this relationship. In addition to this, we must emphasize especially that when he has ascertained this relationship on the basis of an understanding of the transformation of the objects in the aggregates, he must then completely reject what would seem to be the logical operation with the objects -- union -- and construct his model *by dividing* the aggregates of objects, submitting exclusively to the logic of the relationship of whole and part. Thus in the fourth problem variant, if we take the ideal case, the model-making sequence will be the exact opposite of the sequence in which the known values are presented, while the character of the relationship established between the aggregates in the model-making will be the oppostie of the one stated in the verbal description. Obviously, this problem variant would present the greatest difficulty for children.

The fifth problem variant, like the fourth, can be solved in two completely different ways. If the child has already grasped the relationship between whole and part and can subordinate his model-making to it, he can solve the fifth variant in precisely the same way as the fourth (following the second method). But if the child has not mastered this method, he can solve it another way, and the second way of solving this variant turns out to be easier than the comparable way of solving the fourth variant. One could say that this method is suggested by the very sequence in which the numerical values are presented in this variant, as was not the case in the fourth variant. When the child hears or reads the problem, he forms an aggregate corresponding to the first number, and then he is given a second number which characterizes the whole, and along with it the information that this second number was obtained as a supplement to the first. Therefore, stimulated by the conditions of the problem, the child can simply continue counting on up to the second number and then quite naturally count up this supplement. In the fourth problem variant, as we already indicated, it was possible to operate in the same way, but there this mode of action conflicted with the description of the transformations of the objects. A reinterpretation was needed, a transformation, in fact, of approximately this type: "If we say that such-and-such a quantity was supplemented by another, then this is equivalent to saying that this other one was supplemented by the first."

Table 1

Problem Variants for Solution by Making a Model

Variants	Sequence in which Numerical Values are Presented	Sequence of Model-Making	Character of Transformation of Objects	Character of Transformations in Model-Making
1	→	→	⋎	⋎
2	→	→	∧	∧
3	→	→	∧	∧
4-I	→	→	⋎	⋏
4-II	→	←	⋎	∧
5-I	→	←	⋎	∧
5-II	→	→	⋎	⋎
6	→	→	∧	⋎
7	→	←	not indicated	∧

In the seventh variant there is no indication of the character of the transformation of the objects in the aggregates being described, and therefore when models are being made, operations can be chosen only with the aid and on the basis of the concept of whole and part. For children who have not formed this concept it should present considerable difficulty.

The results of the theoretical analysis carried out above are presented in Table 1. The "arrow" represents the direction of the sequence in which the numerical values are presented and in which the models of them are made. The inverted symbol of union in variant 4-I indicates that in that instance this correspondence between the operations of transforming objects and of making models of them is attained through a certain transformation in the meaning.

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AN ATTEMPT AT AN EXPERIMENTAL INVESTIGATION OF PSYCHOLOGICAL REGULARITY IN LEARNING

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Report I:

The Analysis of a Practical Situation From the Standpoint of an Assumed Psychological Regularity.**

Even the simple observation of practical situations can lead to the identification of certain psychological regularities. It is not unusual that, under certain conditions, a person will succeed at an activity. If the conditions are changed slightly, however, difficulties arise and he will make mistakes or perhaps even fail in the very same activity.

Therefore, pedagogical and psychological investigations of incorrect as well as correct operations are completely valid. A recent and most thorough comparative analysis of correct and incorrect operations (and their underlying associations), in conformity with instructional conditions, was undertaken in a monograph by Shevarev [18]. We followed this same method of comparative analysis in an attempt to understand the causes of the so-called "switching errors" observed by Kudryavtsev in his detailed experimental investigation [11].

Kudryavtsev studied the peculiarities of the transition from one arithmetical operation to another, for example, from addition to subtraction and _vice versa_. Pupils in the first through fourth grades were tested individually (the first graders were tested at the end of the school year.

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**Translated by Patricia A. Kolb.

The problems assigned for solution did not in themselves present difficulties for the pupils. At the time of the testing, the pupils already possessed firm calculating skills and rarely made mistakes when they solved such problems under normal conditions. But the author slightly altered the conditions: He presented the same problems in an unaccustomed sequence, and, as a result, the number of mistakes rose sharply. The usual conditions had been changed only in that after a large number of problems of one type, two examples of another type appeared unexpectedly. For example, seven addition problems of the type 6 + 2 = , 5 + 4 = , were followed by two subtraction problems such as 7 - 3 = , 5 - 2 = . Thus, the only peculiarity in the situation was that a "supercharging" of addition operations preceded the subtraction examples. Many pupils did not switch to the new operation; instead of subtracting, they continued to add the numbers. Let us call such mistakes "non-switching errors" (in his book, Kudryavtsev called them "switching errors").

Many experiments in pedagogical psychology have confirmed the fact that non-switching errors are widespread in practical instruction [7, 12, 13, 20, and others] . The school syllabus for arithmetic correctly notes that equipping pupils with sound calculating skills is the most important task in arithmetic instruction [4]. The significance of this requirement has increased greatly in connection with the current school reorganizatin and drawing of the school closer to life and to the pupils' solution of practical problems. Unfortunately, in speaking of calculating skills, one usually overlooks mistakes in operation signs. Suppose a pupil solves the problem 7 - 2 = by giving an answer of 9. He obviously has a sound calculating ability to add 7 and 2, but he does not consider all the peculiarities of the situation and therefore does not perform the operation required for solution. Such "sign" mistakes occur most often when there has been a change in certain of the usual calculating conditions, and are manifested particularly in non-switching errors. It is well known that sign errors remain rather frequent in algebra instruction. The pupils' carelessness alone is an inadequate explanation for the occurrence of these mistakes. It is apparent that there are serious methodological shortcomings in the formation of calculating abilities and skills.

A number of assumptions have been made in the literature about the causes of non-switching errors. Various authors suggesting explanations for these errors have proceeded, consciously or unconsciously, from the comparison of two types of situations. In one of these, non-switching errors occur and in the other they are almost absent (the switch is made). Comparing the two situations, the investigators decided that the frequency of the first of the two operations caused the mistakes. If the first operation was not repeated a sufficient number of times--for example, if both operations were alternated from the very beginning--there would be no (or almost no) mistakes. But no unified opinion exists in the literature about the more profound causes for the errors.

Some authors see the formation of a direction in the completion of the first operation as the cause of non-switching errors (Bzhalava, Khodzhava, Eliava, and others). Others consider the cause to be inertia in the psychic processes, such as thinking (Menchinskaya, Lyublinskaya). Thus, the authors consider different factors to be responsible when an inadequate transition from one situation to another occurs. In the first case, the factor is a lack of narrow direction (that is, broad direction), and in the second, the factors are flexibility and mobility of thinking. Kudryavtsev associates successful switching with the presence of a preliminary analysis of the situations as a whole, as well as with the phenomena of direction. He also observes that non-switching errors may be connected with inertia of the neural processes, and successful switching with their liveliness [11:367].

Unfortunately, not one of the assumptions made in the literature abou the origin of non-switching errors has received vigorous experimental study and can be considered proven. Similarly, the nature of the psychological regularity that conditions successful switching also remains unclear--despite all countervailing tendencies, whether of direction, inertia, or something else.

Through preliminary theoretical analysis it is possible to form a different conception of the causes of non-switching errors and the mechanis of the corresponding correct operations.

In the search for another and possibly more adequate conception, we turned to the works of Asratyan and his followers, who also studied the phenomena of switching [2]. But their test situation could not serve as even a remote model for the situation which interested us--the transition from one arithmetical operation to another. In the first place, a tonic conditioned stimulus (from the test room, from the experimenter, and so on) was used as a signal for switching, and second, there was a fixed stimulus in the background for initiating an operation (the tap of a metronome, a flash of light, and so on). In the experiments of Kudryavtsev, a switch (operation sign "+" or "-") was produced simultaneously with other attributes of the arithmetic problem conditions.

Apparently, it is simplest and most efficient to consider the notation of the conditions of individual arithmetic problems as complex stimuli that prompt defined response operations from the pupils. Each such complex is made up of two basic components--an operation sign and numbers upon which the operation must be performed. Both components essentially influence the operations produced and the final result. That is, each component possesses a known signal activity or "signalness." The occurrence of non-switching errors indicates that the signalness of the sign component has failed. In this situation, the number component can still retain its signalness. For example, suppose a pupil solves the problem 7 - 2 = by responding 9. He adds instead of subtracting, but the result of the addition is correct. In order to clarify the causes of such errors, let us analyze the situation in which the "supercharging" of an operation (such as addition) precedes examples in which the pupils substitute that operation for another (in this case, addition for subtraction). In the case of the "supercharging" of addition, the numbers change but the sign is always the same (for example: 4 + 3 , 2 + 5 = , and so on). The numerical result--the answer--also changed in accordance with the change of numbers in the conditions of the problems. Consequently, for successive complexes of stimuli with varying number and identical sign components, differing response operations are required. This is the type of situation that preceded the errors. Hence our assumption: The given situation causes the subsequent non-switching errors. It is apparent that the gradual *decline of the signalness of the constant sign component* underlies the errors. A question arises: Does the constancy of a component in similar conditions always lead to a decline in its signalness? In other words, how *regular* is the change in the signalness of components in the conditions indicated?

The investigation of Kudryavtsev does not prove that non-switching errors have a regular connection with any defined cause or even a general situation. His work mentions the supercharging of one operation--and the possible formation of a direction to always perform that same operation--as the causes of subsequent mistakes. Yet these same errors are sometimes made without supercharging, and even occur in cases where two operations are intermittently counterposed [11:384] . It is clear that without the complete statistical processing of data from appropriately-constructed experiments, it is impossible to prove the presence of a regular connection.

The practical significance of the question posed above becomes clear when one considers the peculiarities in the construction of arithmetic texts and problem books for the first grade [15]. In the introduction of almost every new operation, the authors consciously avoid presenting it simultaneously and in contrast with a similar or opposite operation. For example, in the study of the addition and subtraction of numbers between 1 and 10, the exercises are all arranged so that the addition of a certain number is treated first, while the subtraction of the same number and the contrast between the two operations are introduced only in succeeding lessons.

If our assumption about the origin of non-switching errors is correct, then it must be recognized that the textbook authors create conditions from the very beginning conducive to a reduction in the signalness of operation signs, and that they then attempt to correct the situation as far as possible. The widespread occurrence of non-switching errors is eloquent testimony to the fact that, in the end, they do not succeed.

Tests aimed at *overcoming* non-switching errors and organized in a different pattern (for example: 4 + 2 = 6, 4 - 2 = 2) are the most adequate check for the existence of the regular relationship (formulated above) between the signalness of a problem's sign component and its invariance. In these tests, it is no longer the number component that is varied, but instead, the sign component. As before, successive complexes of stimuli will demand different operations. Since the only distinguishing feature of the complexes will now be the sign component, one might anticipate an increase in its signalness. If problems are then given according to the first pattern, with a supercharging of addition, the increased signalness of the sign component should result in a decrease in non-switching errors. The development of an appropriate methodology for the tests will be the theme of the next report. Positive results to tests constructed according to this methodology would strengthen the case for the regularity that we have assumed in the conditions of arithmetic instruction.

In our first work [5, 10] , the regularity under consideration served as the basis for one of the most effective methods of strengthening the weaker component of complex stimuli. With this method, various reactions were developed to complexes of stimuli with identical strong components and different weak ones.[1] In two other works [7, 9], we attempted to trace the same regularity in conditions of elementary geometry instruction.[2] The choice of situations studied in all three of these works was not accidental. We were interested in manifestations of the desired regularity along two parameters and in two basic types of situations: 1) when the features of the complex stimuli were parts or properties of the complexes (an example of a part is a table leg; examples of properties are the color or shape of the table), and 2) when the test was conducted with the presentation of particularly artificial stimuli, or under more natural conditions--

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[1] The method of "leveling" components proposed by Vatsuro [19] might be considered to be a particular case of this method. In such "leveling," the complexes are also distinguished only by weak components: A positive reaction is produced to one complex, an inhibited reaction to the other. For a more detailed evaluation of this method, see another of our works [7].

[2] Shevarev has proposed the following letter scheme for designating the situations in which the indicated regularity can occur: AF→R; AG→S; AN→U, where A is the constant component of the complex stimuli; F, G, and H are changing components; and R, S, and U are different reactions to the complexes [18:169].

such as those in geometry instruction. In the works mentioned, manifestations of a regularity in conditions of instruction were not studied when the features of the situation constituted its parts. Situations of solving arithmetic problems whose sign and number features are its parts (and which features, therefore, can be easily distinguished from one another) present a convenient case for this type of study. Another peculiarity of the arithmetic problem situation compelling its special study is the particular nature of its features. In our investigations in geometry [7, 9], one of the features of the situation did not have essential significance, and it was not necessary to determine the examinees' operations if these operations were correct. (For example, in evaluating the perpendicular relationship of lines, their position on the plane is not essential if the angle between them always remains a right angle.) In arithmetic examples, however, the two basic features of sign and number are both essential.

Report II:

On the Varied Use of the Regularity of Differentiating Complex Objects for the Elimination of Sign Errors in Arithmetic Problems.*

In the previous Report we discussed the following regularity: If the complex objects perceived by a person are identical in some features but different in others, and reactions to these objects are developed or strengthened, then the identical features either become non-signalling or their signalness[3] decreases. For brevity, let us designate this as "regularity A."

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*Translated by Patricia A. Kolb.

[3] We shall say that a certain feature has signalness (or valency) for the person if reactions somehow depend on the presence or absence of this feature.

It must be assumed that in pedagogical practice regularity A can have positive or negative consequences--wherein one or another of its effects may be determined by certain supplementary conditions. The purpose of our investigation was to test this assumption. This report presents the first part of the completed investigation.

Four series of experiments were conducted with first- and second-grade pupils. The first graders were tested at the end of the school year, the second graders at the beginning. All the series employed addition and multiplication problems of the types 4 + 3 = and 6 x 2 = . The problems were not difficult for the pupils and they solved them correctly. Difficulties arose and mistakes sharply increased only under certain conditions especially created in our experiments. We shall concentrate on the basic features in each problem--sign and number.

Study I - Background Experiments

Let us agree to call the first experiment of this study the basic experiment. In it, seven addition problems (with sums not exceeding 10) were interrupted by two multiplication problems (also very simple--with the answer 12), followed by one more addition problem.

The basic experiment was conducted with 141 pupils in four different classes of three schools in Moscow. Each pupil received a prepared card on which the indicated 10 problems were arranged in a column. The first assignment was to "copy the column of problems on your clear card." This task was done at the beginning of the arithmetic lesson and took from ten to twelve minutes. The cards completed by the pupils were then collected. The second assignment, which was given at the end of the lesson, was to "solve all the problems in order and write the answers on the experimenter's card." To some degree it was thus possible to separate mistakes in the copying of signs from sign errors in the operations--or answers. An example of a sign error in the answer is "4 x 3 = 7." In processing the results, we were interested primarily in non-switching errors, that is, sign errors in the signs and answers of the last three examples in the column. In a second experiment on the following day (let us agree to call it the control experiment), no special changes were made in its conditions, and all of the conditions of the first experiment from the previous day were repeated as exactly as possible.

The transition to other arithmetical operations in the column's last three problems caused a significant number of the pupils to make non-switching sign errors in both the basic and the control experiments. From 6.6 to 7.6 percent of all the signs copied by the pupils in the indicated problems were copied incorrectly (discounting whether the errors were corrected or not). The corresponding percentages of mistakes in writing down answers varied from 7.8 to 9.9 percent in different experiments (see Tables 1 and 2). A comparison of all the mistakes made by the examinees in the two consecutive experiments--the basic and the control--showed that no statistically reliable change in the number of sign errors made in copying and in "operations" (answers) was observed (see Tables 1 and 2).

Table 1

Correctness of Signs For Basic and Control Experiments in Four Studies

	Study			
	I	II	III	IV
Total cases.	423	303	567	483
Percentage of mistakes in the basic experiment	7.6	8.9	10.2	8.7
Percentage of mistakes in the control experiment . . .	6.6	10.9	5.1	3.5
Difference in percentage of the mistakes in the basic and the control experiments. .	+1.0	-2.0	+5.1	+5.2
Reliability coefficient* of the difference between the results of the basic and the control experiments. . . .	1.1	0.8	3.2	3.5

*This is the ratio of difference to the standard error of the difference--a _t_ statistic (Ed.).

Table 2

Correctness of Answers for Basic and Control Experiments in Four Studies

	Study			
	I	II	III	IV
Total Cases	423	303	567	483
Percentage of mistakes in the basic experiment.	9.9	8.6	13.6	14.1
Percentage of mistakes in the control experiment . . .	7.8	11.9	13.9	5.6
Difference in percentage of the mistakes of the basic and the control experiments	+2.1	-3.3	-0.3	+8.5
Reliability coefficient of the difference between the results of the basic and the control experiments	1.1	1.3	0.2	4.5

The results obtained in each of the two experiments in the first study were in complete conformity with regularity A. The operation sign (+) remained the same in the first seven problems; only the numbers changed. Therefore, the operation sign (the sign feature of the problem) lost signalness for some pupils. Only the numbers, which varied, retained signal value. As a result, when these pupils encountered the last three problems they made sign errors in copying or in the answer. Here we observe the negative influence of regularity A in the completion of a school assignment. The nature of regularity A's usage completely determined this negative effect: All the essential criteria were not varied in the first seven problems in the column (the operation sign was constant).

Studies II, III, and IV.

In the three succeeding studies the plan described for the two experiments remained unchanged except that a training experiment was conducted between these experiments (usually immediately before the control experiment and during the same lesson)--the pupils had to copy and solve some "practice" problems. The results of the solution of the practice problems were immediately checked and corrected. All the succeeding studies were conducted each time with new examinees in different class groups. There was no preliminary selection of whole groups or of individual examinees.

In Study II, the 10 practice problems were of the type 5 x 2 = and 9 x 1 = . Only the numbers were varied in the practice problems of this series. In comparison with the basic experiment, the sign had changed--that is, a multiplication rather than an addition sign was constant. (Such sequential contraposing of different operations is widely employed in the first-grade arithmetic textbook by Pchelko and Polyak [15]. The experiments with 101 examinees (from 3 new classes) show the results of this.) Neither mistakes in the writing down of signs nor sign errors in the answers decreased in the control experiment when compared with the basic experiment. On the contrary, there was some tendency toward an increase in both types of mistakes (although statistically this was not completely reliable--see Tables 1 and 2).

It could be conjectured that the positive influence of regularity A would be manifested in the conditions of Study II. In the basic experiment of this study the pupils dealt with 8 addition problems, and in the training experiment, with 10 problems in another operation--multiplication. In other words, there was a situation of sequential contraposing. The problems in the basic and in the training experiments differed only in essential features (signs and numbers). Consequently, it could be expected that the signalness of the operation signs would increase after the basic and the training experiments, and that the number of both types of sign errors would, in turn, decrease in the control experiment. Actually, as stated above, mistakes did not at all decrease in the control experiment, and even had some tendency to increase.

Apparently, there were certain additional conditions operating in the described experiments that disguised the anticipated positive influence of regularity A. The age peculiarities of our examinees might have been such supplementary conditions. They were primary-school pupils for whom sequential contraposing did not facilitate the differentiation of problems according to their essential features, and likewise did not ensure the positive influence of regularity A. In order to test the role of the indicated factors in Studies III and IV, we introduced intermittent contraposing.

Ten practice problems of the type 4 + 2 = and 4 x 2 = , were used in Study III. Here the numbers were constant and only the signs varied. The sequential order of the signs ensured intermittent contraposition (+, x, x, +, x, +, +, x). There were 189 examinees in six class groups. In the control work there was a decrease in the number of mistakes in copying signs, but no change in the number of incorrect answers. This conclusion was statistically reliable (see Tables 1 and 2).

In 10 practice problems in Study IV, both of the essential problem features were varied: the signs and the numbers. For example: 4 + 2 = , 7 x 1 = . There were 161 examinees in five classes. A comparison of the basic and the control experiments showed a statistically reliable decrease in the number of both types of sign errors in the control experiment (see Tables 1 and 2).

The results of Studies III and IV fully confirm the influence of regularity A on young schoolchildren in the conditions of intermittent contraposition. Correctness in copying signs presupposes the signalness of the sign feature. We varied the operation signs in both series, which caused their signalness to increase for a number of examinees and led to a regular decrease in the number of mistakes in copying the signs. Writing down answers correctly further required the signalness of both essential problem features--sign and number. Therefore, when we used regularity A to increase the signalness of only the number feature (as in Studies I and II), or of only the sign feature (as in Study III), it did not result in a decreased number of incorrect answers. As we would then expect, this positive effect was observed in Study IV under conditions where the problems were immediately differentiated according to all (two) of the essential features.

Our research has thus shown with statistical reliability that, first and foremost, regularity A has significance in one area--elementary-school arithmetic instruction. But we can also now assume that this regularity has broad significance and is relevant to the most varied areas of human activity. This research has further demonstrated that certain supplementary conditions can have essential significance. In particular, both the positive and the negative influence of regularity A--as well as its lack of influence--on the pupils' completion of a school assignment were observed to depend on such conditions in our experiments. The results we obtained in these experiments depended on such variable supplementary conditions as:

1) The choice of perceived objects ("problems")
    a) which individual, essential features of the objects being differentiated we made different or identical;
    b) whether or not the objects being differentiated differed in all essential features.
2) The temporal relationships between the objects being presented: the sequential or intermittent contraposing of these objects.

Certain variable conditions in our work were only outlined. The sign and number features of the problems in our experiments were separable (they could be separated from one another). In our earlier work on schoolchildren's mastery of elementary geometrical knowledge [7, 9], the varying and the constant features of the geometric features--their form and their spatial position--were inseparable.

Thus, we are able to ascertain the effectiveness of regularity A in changing the signalness of both separable and inseparable features. Further study of this regularity will be required in order to identify its specific manifestation(s) under different conditions, with separable and with inseparable features. Deeper study of all of the conditions named--and possibly of the many other variable ones as well--is necessary. Without such an accounting, the expedient use of regularity A for practical goals will be impossible.

Report III:

The Use of the Regularity of Differentiating Complex Objects in the Teaching of Arithmetic Operations with Sums Less than 10.*

In the previous report involving experiments with pupils from grades 1 and 2, a regularity was verified and tentatively designated "regularity A." We studied that regularity primarily in the strengthening of differentiations that had been elaborated in the past. Under those circumstances the signalness of identical signs decreased while the signalness of different signs increased. Our current work will include the study of regularity A in the conditions of the formation of new connections. If our earlier work was directed at clarifying methods of removing pupils' errors, our present task will be to prevent errors from arising from the very beginning by finding methods of using regularity A in the formation of new connections.

In analyzing the textbook in arithmetic for grade 1 [15], as well as methodological handbooks for teachers [1, 17, and others], we observed that an obvious preference is given in both the textbook and the handbooks to the method of sequential contraposition. Examples and problems are selected there in an appropriate fashion.[4] Thus, in the textbooks's introduction of two opposite operations (addition and subtraction, multiplication and division), exercises are first given on working out one operation without the other. As a rule, the illustrative lesson units in the methodological handbooks follow the textbook. These operations in the coverage of each new topic (adding and taking away 1, 2, 3, and so on) are not only introduced separately, but also are introduced in different lessons from the very beginning of systematic study of the section called "Addition and Subtraction." The gap between different operations is subsequently increased still more by the appearance in the textbook of entire topics--consisting of several lessons each under the immediate headings of "Addition without Passing through the Bounds of Sums Less Than Ten" [15:65], "Increasing by Several Units" [15:67], and so on. Thus, from the outset, there is lacking the differentiation of problems on addition and subtraction, which could have been distinguished according to all essential features (they should be distinguished not only by numbers, but also by operation signs). Only after one operation on a certain topic has been reinforced are exercises on the other operation and intermittent exercises on both operations introduced, with many fewer lessons allotted to the latter than were spent on the individual study of one operation or another.

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*Translated by Harvey Edelberg.

[4] In this report we will not set forth the results of our investigation into the application of regularity A to instruction in problem solving.

Apparently, under the influence of textbooks and methodological handbooks, there is a widespread fear among methodologists and elementary-school teachers that the "simultaneous" (intermittent) introduction at the very beginning of opposite and generally different operations can only mix up the pupils, give rise to the confusion of different operations, and make difficult the whole process of learning. One cannot consider this judgement alone, but must subject it to experimental testing under the actual conditions of instruction.

In the previous investigation conducted with pupils in grades 1 and 2, it was shown that the intermittent contraposing of arithmetic problems requiring different operations better promoted the isolation of features necessary in the perception and solution of these problems than did sequential contraposing. One can assume that the indicated advantage of intermittent contraposition is also preserved in the formation of new connections. In sequential contraposition two series of objects (for example, two columns of problems) are presented. Within each series the objects do not differ in all of their essential features (for example, they have identical signs); only the objects of the two different series differ at once from one another in all essential features (for example, in both number and sign). Apparently, the reason for the ineffectiveness of sequential contraposing lies in the following two peculiarities of regularity A when it is manifested under such contraposing: 1) the objects of one and the same series are identical, even if only in one feature. Therefore, regularity A must act here in a negative direction--identical features may become non-signalling--and 2) the objects of different sequential series are distinguished by all essential features. This must foster a positive manifestation of regularity A in the relationship of all of these features, although the time interval between the objects of the different series does hinder this. The latter interval is significantly greater, on an average, than the interval between objects of one and the same series, and it is possible for this reason that the positive effect of regularity A masks the negative effect (see the first peculiarity enumerated above). Our task consists of verifying our hypothesis concerning the advantages of intermittent contraposition in the formation of those new connections that form the basis of calculating operations within the limitations of the natural numbers with sums less than ten.

Experimental instruction was organized in grade 1-b of School Number 672 in Moscow. The control grades were 1-a and 1-c of the same school.[5] Until the tenth of October, 1960, all three grades were instructed strictly according to the textbook and the teaching methods in general use--which were adapted to that textbook. On October 10 we conducted background control work in all three classes, before going on to the systematic section entitled "Addition and Subtraction." The children had to copy and solve a column of five problems: In the first three, they had to add 1; in the fourth, subtract 1; and in the fifth, again add 1. All of these operations were already well known to the pupils. Thus, the summary errors they made in the answers did not interest us nearly as much as those which depended on the incorrect copying of signs. According to the number of sign-copying errors made in working the column's last two problems, grade 1-b made the greatest number of errors. Grade 1-a made three mistakes, 1-c made five mistakes, and grade 1-b made eight mistakes. (Relative to the total number of observations made in each class, the percentages were 4.8, 9.3, and 12.1, respectively.) Since the number of errors was greatest in 1-b, that grade was chosen to be the experimental class. In what follows, the control classes were instructed, as before, according to the textbook. They were used as a basis for evaluating the effectiveness of the method of sequential contraposing. In the experimental class, addition and subtraction were studied "simultaneously" from the very beginning or, more precisely, through the method of intermittent contraposition. The results were used to evaluate the effectiveness of our earlier introduction to intermittent contraposition.

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[5] I express my deep thanks to the school's Director, S. G. Amirjanov, and to the following teachers for their great help in conducting the investigation: T. N. Akhapkina, L. V. Maksimova, and N. I. Titova.

In order to achieve a better realization of the principle of intermittent contraposing, we decided to start from a logical structure of numbers. We shall take as an example-problem the number 3. The children successively mastered the knowledge of all cases of addition and subtraction within the limits of this number by first taking real objects, effecting real operations with these objects, and then considering them in the abstract.[6] $2 + 1 = 3$, $3 - 1 = 2$ (this latter operation was introduced immediately as the inverse of the first), $1 + 2 = 3$, and $3 - 2 - 1$. Different operations in the study of the structure of the remaining numbers in the first ten were contraposed in an analogous manner. (It is possible to represent the logic of our consideration of all the cases by the set of formulas: $x + y = a$, $a - y = x$; $y + x = a$, and $a - x = y$ --where x and y assume, consecutively, all values from 1 to 9, and $a \leq 10$. In each successive topic, the quantity a increases by 1.) As a result, a table of addition and subtraction for the numbers 2 to 10, which the children understood without particular difficulty, was put together step by step.[7]

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[6] Here it is important to emphasize that such general principles as "from the graphically active to the abstract," and others, were not objects of special study; we therefore tried to put them into practice in equal measure in both the control classes and the experimental class.

[7] It is true that such instruction logic almost completely excluded the possibility of using the textbook for home (and class) assignments which, of course, gave known advantages to the control classes. We were not able to use even those sections of the textbook in which examples or problems on both operations were alternated. However, the advantages originating for the control classes in all other relations except the one being studied (the use of regularity A), could only increase the reliability of the results of our investigation.

Calculating exercises for each topic were set up in the textbook according to the principle of sequential contraposition, as indicated above; but this, in turn, excluded the possibility of proceeding from the logic of the number structure. In fact, the order of studying operations in the textbook is reflected in such headings as "Add and Subtract 2" (at first only add, and so on), "Add and Subtract 3," and so on. According to this order the subtrahend and second addend remain constant each time, and all of the remaining components (numbers and signs) change. (It is possible to generalize the given principle in the form of two formulas: $x + a = y$ and $y - a = x$, where x and a change from 1 to 9 and $y \leq 10$. In each successive topic the quantity a increases by 1.)

The following question may arise: Does not this study of number structure constitute that supplementary factor by which experimental instruction was distinguished from instruction in the control classes, and which thereby caused the difference in results? Here one must bear in mind the fact that the study of number structure is also given a great deal of attention in the ordinary teaching methodology. Specifically, the pupils in our control classes answered questions about the composition of these or those numbers with as much success as the pupils in the experimental class. The difference consisted mainly in the fact that in the control classes the structure of numbers was studied primarily in connection with the synthesis of numbers by means of addition, while in the experimental class this same composition was revealed equally by both synthesis and analysis, by addition and by subtraction. As we can see, the possible influence of the indicated supplementary factor apparently boils down to that basic difference in ways of combining the contrasting operations (which has also served as the fundamental subject of our investigation). That is, it reduces to the difference between sequential and intermittent contraposing.

Approximately two months after the new instruction in the experimental class, three identical control tasks were introduced in all three classes (with two variants per class). In each class these tasks were conducted at the time of transition to the study of the second ten natural numbers;

that is, roughly at the same time in all three classes. In the first control work the pupils had to copy and solve three columns of five problems each. The problems were arranged alternately in two operations (in random order, but with no more than two repetitions of one operation--either addition or subtraction). Problems on all types of operations--within the limits of the natural numbers up to ten--were used, including those with the answer 0.

In the second control task the examples in columns two and three were more complex trinomial problems (of the type 10 - 2 - 5 = ?). Table 3 shows the difference in results between the experimental and normal methods of instruction in both control tasks.[8] This difference

Table 3

Errors in the Answers to the Problems of the Two Control Tasks

Results	1st Control Task		2nd Control Task	
	Experimental Class	Control Class	Experimental Class	Control Class
Total number of problems solved	495	915	384	660
Percentage of incorrect solutions	3.6	7.5	7.8	13.6
The difference in percentages between the quantity of mistakes in the experimental class and that of the control classes	3.9		5.8	
The coefficient of reliability of this difference	3.3		3.1	

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[8] In the two control classes the percentages of incorrect solutions were roughly equal to one another, and were always higher than in the experimental class. This allowed us to combine the results from the two control classes in our reliability computation.

in favor of the experimental instruction was completely reliable statistically (the coefficient of reliability, that is, the ratio of the difference in results to the mean error of this difference, exceeded 2.6). In the general mass of incorrect answers the number of sign errors in the answers was negligible: In both the tasks the control-class pupils committed 7 errors, while the experimental-class students did not make a single mistake--even a corrected one. Meanwhile, we could apparently judge more simply the merits of different kinds of contraposing of arithmetic operations by the ratio of sign errors in the classes. A third control task (as always, with two variants per class) was conducted for the purposes of further selecting appropriate facts. The pupils had to copy from the board and solve three columns of normal, binomial problems--with five problems per column. The difference between this (the third) and the first two control tasks consisted in the fact that here addition was repeated four times in the first column, and then was followed by a last problem (in the same column) in subtraction. As was shown in Report II, this device promotes a negative manifestation of regularity A. In the given case it was bound to help show the stability of the differentiation of various operations under different methods of instruction. The superiority of the experimental class appears particularly distinctly against a background of the very first control task--mentioned above as background control work, and conducted in all classes before experimental instruction was begun, and used to select the experimental and control classes. The experimental class went from last to first place in the number of mistakes made in copying signs (see Table 4). The difference in results in this last task between the experimental and the control classes is statistically completely reliable (see Table 4). One can add that all of the errors in the experimental class (and there were 3 of them) were corrected by the pupils themselves, while in the control classes precisely half of all the errors were uncorrected ones (17 and 17--34 in all).

Included in the examples of the third control task was a special addition problem containing a transition past the sums less than ten (for example, 8 + 3). The successful solution of such a problem during

Table 4

Sign Errors in Copying Problems Before and After Various Methods of Instruction

	Control Work 10/10/60		Control Work 10/13/60	
	Experimental Class	Control Class	Experimental Class	Control Class
Total number of problems solved	66	116	450	960
Cases of sign errors in copying (in percentages of the general number of problems)	12.1	6.9	0.7	3.5
The difference in percentages between the quantity of mistakes in the experital class and that of the control classes	12.1 - 6.9 = 5.2		0-7 - 3.5 = 2.8	
The coefficient of reliability of this difference	1.1		3.9	

this period when the classes had yet to begin studying the second ten natural numbers and their corresponding operations, served as an indicator of the pupils' degree of auxiliary preparation at home. In this case it turned out that the experimental grade had no advantages over the control classes. The number of incorrect solutions and refusals to complete the problems proved to be identical (9) in all classes. From this point on it was possible to judge, to a certain degree, that the difference in fundamental results was not determined by a difference in home preparation between the experimental and control grades (which, in any case, would have been unlikely before checking), but depended instead, on the different use of regularity A in these classes.

Thus, we also observed the manifestation of regularity A in the development of new connections on complex objects (as before, we saw it here in terms of the strengthening of connections). Moreover, intermittent contraposing better promoted the development of correct and strong connections than did sequential contraposing.

Report IV:

The Relative Difficulty for Pupils of Three Types of Indirect Problems on Addition and Subtraction*

With this report we begin a presentation of the results of an investigation of the regularity of the differentiation of complex objects (regularity A)[9] under conditions accompanying the solution of arithmetic problems expressed in oblique form (more briefly, oblique, or indirect problems). The choice of these conditions was determined by two series of considerations.

First, these problems have peculiarities not taken into account in the earlier studies of regularity A:

a) the differentiated objects in this study are characterized by a greater complexity, in view of which even the essence of some particular features, i.e., their connection with the required operations, is not entirely obvious;

b) the objects are completely or partly presented in verbal form;

c) the pupils' response activity is significantly more complex;

d) the degree of the distinction of objects according to a given feature is varied.

Second, it is common knowledge that these problems are of great difficulty for the pupils. As early as 1950 they were completely removed from the first-grade curriculum. At present some indirect problems are studied in the second grade, e.g., problems on finding the minuend or addend. Even here, however, matters are unsatisfactory [14]. The existing methods of teaching how to solve indirect problems undoubtedly need perfection. Until now we have had no psychologically based answer to questions of such primary importance to school practice as:

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*Translated by David A. Henderson

[9] Regularity A may be formulated briefly as: If the complex objects perceived by a man are identical in some features and different in others, and various reactions are redeveloped or strengthened in these objects, then the different features acquire predominant significance (for more details see Report III).

a) What is the relative value of the method of contraposition among the other methods of singling out the features of arithmetic problems?

b) What types of indirect problems is it expedient to teach first?

c) When should the contraposition of direct and indirect problems be introduced--before consolidating abilities to solve direct problems or at the very time such abilities are being developed?

d) What should be the form of concrete methodological devices for successful instruction in the solution of indirect problems as opposed to direct problems?

We assumed that consideration of regularity A was of no little importance in answering these questions, at least in the first approximation. Also, in answering these questions by experimental means, we hope to ascertain the peculiarities of the phenomena of regularity A in the new circumstances indicated above.

Before approaching the study of these four questions, we should do some preliminary work on the classification of indirect problems in their connection with direct ones. The major goal (and the basic difficulty) of such work consists in finding the objective criteria, features, on which a classification can be founded. Then we must ascertain, at least hypothetically, how the various types of problems relate in difficulty for the pupils. This is necessary to establish any sequence for teaching the devices for solving problems of the various types. Such are the main tasks of the present report.

To determine the general principle for the classification of indirect problems we can limit outselves to problems on addition and subtraction. Problems on the other operations may easily be included in the common scheme of problem types that is obtained by the classification of indirect addition and subtraction problems.

All varieties of simple direct and indirect addition and subtraction problems can be solved identically in one of two ways:

I. $A + B = ?$ (S)

II. $S - A = ?$ (B) or $S - B = ?$ (A)

where S (sum indicates the "whole" total number, and A and B are parts of this whole. The two plans for solving problems (number problems) represent, essentially, the notation of two algorithms of addition and subtraction problems. But in the framework of these algorithms it is still impossible to differentiate the direct and the indirect problem. The specific features of these problems lie outside their common algorithms; these features come forth when the actual texts of direct problems are compared with each other and when those of indirect problems are compared with each other, on the one hand, and also when the texts of direct problems are compared with texts of corresponding indirect problems, on the other.

Let us compare generalized texts of all the basic variants of direct and indirect problems, e.g., of the type:

1. Direct problems:
    a) There were A objects of the first kind. There were B more of the second kind than of the first kind. How many objects of the second kind were there?
    b) There were S objects of the first kind. There were B fewer objects of the second kind than of the first kind. How many objects of the second kind were there?
2. Indirect problems:
    a) There were S objects of the first kind. There were B more objects of the first kind than of the second kind. How many objects of the second kind were there?
    b) There were A objects of the first kind. There were B fewer objects of the first kind than of the second kind. How many objects of the second kind were there?

The concrete formulas for these problems will be:

1. a) A + B = ? (S); b) S - B = ? (A)

2. a) ? (A) + B = S; (b) ? (S) - B = A.

As we see, the direct problems differ in only one component.[10] This component is the word more or fewer. Similarly, indirect problems differ in only one component. This is the same two words, more and fewer. Let us call this component distinguishing the direct and indirect problems given above Distinctive Feature I. In our example this feature has two variants (the words more and fewer).

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[10] Here S always denotes the whole, and A and B denote its parts. At the same time, different literal designations of data in the texts of problems (A, B, S, B) do not denote differences in the numerical quantities of different problems. In particular, A in one of the problems can denote the same quantity as S in another of these problems.

A second essential feature of direct problems is the feature that allows the distinction of an indirect problem from the corresponding one.[11] Let us agree to call it the distinctive feature of the direct and the indirect problem, Feature II, for short. In our example (see the formulas for the four problems, above) Feature II will be the position of the unknown in the problem's structure.

Here one variant of Feature II--the unknown--is in the unsigned part of the equation; this variant characterizes direct problems. The second variant of the same feature objectively characterizes indirect problems. The unknown stands in the signed part of the equation.

In solving a previously unknown problem, be it direct or indirect, the pupil's operations should be determined by both features, I and II. Let us call this set of the two features (rather, of variants of these features) a complex feature.

Using Features I and II then, we obtain the following classification of simple arithmetic problems on addition and subtraction:

Direct problems

Formulas: 1) A + B = ? (S); 2) S - B = ? (A). The unknown is in the unsigned part of the equation. Feature I can be distinctly expressed in a verbal formulation of the problem owing to special lexical units: more, fewer, added, took away, was obtained. In the actual texts of problems all these words may be represented by their equivalents: more = more expensive, higher, longer, . . .; added = flew together, gave more, . . .; was obtained = was, remained, was altogether, etc. In the following formulation Feature I is lacking: "There were A objects of one kind and B objects of another kind. How many objects of the two kinds were there?"

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[11] We will call problems that are not differentiated by Feature I "corresponding" direct and indirect problems.

Indirect Problems

The unknown in the formulas of these problems is in the signed part of equations.

Type I--indirect problems. Formula: $B + ?\ (A) = S$. Feature I is absent from the verbal formulation of the problem. One possible variant of the formulation is: "There were S objects of two kinds. There were B objects of one kind. How many objects of the other kind were there?" Such problems may be transformed into direct problems with the addition of several words--carriers of Feature I ("they took," "the rest," etc.)[12]

Type 2--indirect problems. The second type of indirect problem includes the following variants of formulation (in general terms): a) There were some. When B was added, S was obtained. How many were there? b) There were A. When several were added, S was obtained. How many were added? c) There were several. When B were taken away, A was obtained. How many were there? d) There were S. When several were taken away, B was left. How many were taken away? The concrete formulas of these four problems will be: a) $?\ (A) + B = S$; b) $A + ?\ (B) = S$; c) $?\ (S) - B = A$; d) $S - ?\ (A) = B$.

Type 3--indirect problems. The third type of indirect problem includes problems on increasing and decreasing. The general formulations of the two possible variants of such problems were given above ($?\ (A) + B = S$; $?\ (S) - B = A$).

Since a characterization of the unknown is required for solving the problem, we can transform indirect problems of types II and III into direct problems where this characterization is given, i.e., *turn* the indirect problem into a direct one. As a result of such transformation, instead of the four formulas we obtain the two basic algorithms (given above) for simple addition and subtraction problems (coinciding with the formulas of direct problems): $A + B = ?\ (S);\ S = B = ?\ (A)$.

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[12] The carrier of Feature I in indirect problems of the first type may be real operations with objects. In teaching such problems, the teacher usually accompanies the text of the problems with actual operations with objects, gestures, etc. Independently or with the teacher's help the children name these operations with the appropriate words which, of course, is equivalent to transforming an indirect problem into a direct problem.

The difference between the second and third types of indirect problems consists in: 1) a different number of possible variants (2 and 4); 2) two quantities figure in problems of the third type, and in problems of the second type we are dealing with changes in a single quantity; 3) Features I and II are lexically more fully expressed in problems of the second type. For example, in a problem of the third type there is only one word, the carrier of Feature I (*more* or *less* or their equivalents). At the same time, in problems of the second type there are more such words (in actual texts besides the word *added*, as in our generalized text (see above), words like *there were altogether*, *still*, *remained* are also possible).

Practical school instruction shows that with the existing methods of instruction, indirect problems are more difficult for children than direct ones. This statement scarcely needs further checking. The causes of difficulties arising in children when solving indirect problems derive, we must assume, from the methods of instruction. Existing methods of instruction do not ensure proper utilization of regularity A when the solution of simple problems is being taught. Even in the first half of the first year of instruction there develop in the pupils firm associations between the variants of Feature I and the arithmetic operations being performed: between the words "added," "more," or their equivalents and addition, and between the words "took away," etc., and subtraction. The subjects do not differentiate problems by Feature II. This is understandable because this feature does not have signalness--instead of the complex feature, it is Feature I which wholly determines the children's operations. It is no accident that the most typical and very stable mistake of pupils is that they solve indirect problems as direct ones.[13] The incorrect solution in this case is determined only by the variants of Feature I.

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[13] The tremendous difficulties in overcoming these errors occasioned, unfortunately, the removal of indirect problems of all three types from the first-grade curriculum, and problems of the third type from the elementary school curriculum altogether.

We decided to compare the difficulty of the three types of indirect problems. It may be assumed that problems of the first type should be least difficult and problems of the third type, the most difficult. This assumption is based on the following considerations.

Success in solving all simple problems is determined by the signalness of the complex feature (see above). The variants of Feature I (more--fewer, gave--took, etc.) are opposed to one another. The variants of Feature II ("both quantities in the signed part of the formula are known"--"one quantity is known, the other is not") are not in an opposing relationship; in other words, the variants of Feature I differ from one another more than the variants of Feature II. We conducted earlier laboratory experiments [5, 8] with two complex stimuli whose peculiarities were: a) each of the stimuli, characterized by two features, had one of two variants of one feature and one of two variants of the other feature; b) the variants of the first feature were opposed to one another, the variants of the second feature were relatively little different from one another; c) when shown one of the complex stimuli, the subject was to perform one operation, and when shown the other complex stimuli, another operation; d) the variants of both features were more visual and simpler than variants of Features I and II, with which we are now concerned. Hence, we may consider that the processes occurring in these laboratory experiments are a model of the processes occurring in the solution of arithmetic problems of the types we are now considering. The laboratory experiments showed that under the described conditions the variants of the first feature became dominant, i.e., the subject reacted in some way contingent upon the variants of this feature. The variants of the second feature, however, are recessive, i.e., the pupils' reactions are not contingent upon them. It is possible that the processes of the perception of the first feature impede the processes of the perception of the second feature. Bearing this in mind, one may also presume that, in the solution of the arithmetic problems we are examining the variants of Feature I will be (at least in some pupils) dominant, and the variants of Feature II will recede into the background. But, as we already said, the solution of a problem, correct not only in its results but also in its structure, should be determined by the variants of both the features. Consequently, it is to be expected that either the pupils will err in their solution of problems where the required operation does not correspond to a variant of Feature I (when, for example, the word more is in the conditions, but the true operation is subtraction)or the solution process will be delayed. But Feature I is absent from the conditions of the first type of indirect problems, so the phenomena just described cannot occur. With this as a starting point, one must assume that the first type will be easier, i.e., the pupils will make the least number of errors here.

In the conditions of indirect problems of the third type the difference between the variants of Feature I is greater than in indirect problems of the second type. Hence we may expect indirect problems of the third type to be more difficult for the pupils.

To ascertain the relative difficulty of the three indirect problem types and to check the above stated theoretical proposition of the unequal difficulty of these problems, we conducted two variants of control work for all types of problems. The texts of problems of one variant were:

1) "Nine trees were growing at the entrance of a school, birches and poplars. There were 7 birches. How many poplars were there?"

2a) "Vasya had several acorns. When he planted 3 acorns, he had 6 left. How many acorns did Vasya have to start with?"
[16: problem 224]

2b) "Sasha had 5 stamps. When he was given several more stamps, he had 8 altogether. How many stamps was Sasha given?"

3) "Seryozha had 5 apples. Seryozha had 2 apples more than Misha had. How many apples did Misha have?"

The control experiments were conducted at the end of the first or at the very beginning of the second year of instruction. Before this time, the usual instruction following the curriculum and the workbook had not included a single one of these three types of indirect problems (in the classes we studied).

In working out the results, we considered the percentage of pupils who completely solved the problem correctly. Moreover, to best delimit the different types of problems according to difficulty, we also added together the percentage of pupils with wholly correct solutions and pupils who made one specific error--incorrect notation of the operation.

Here the unknown number was found correctly, but it figured in the notation not as the result (the answer), but as an already known quantity. An example of this kind of error, in the solution of the first of the 4 problems given above is: $7t + 2t = 9t$ (the unknown is $2t$).

Table 5

Relative Success in Solving
the Various Indirect Problems
Before the Special Instruction

Problem Type	Total Number of Pupils Working	Solved Completely Accurately	Correct and Partly Correct Solutions
		In % of the Total Number of Pupils	
1	54	59.3	68.5
2a	44	31.8	59.1
2b	26	7.7	53.8
3	108	9.3	11.1

The results of the control work (see Table 5) confirmed our expectations. The most difficulties arose for pupils when they were solving indirect problems of the third type, and the least, in solving problems of the first type. The statistical elaboration of the results showed the reliability of this statement.

Report V:

Schemes for Solving Direct and Indirect Problems and the Planning of Experimental Instruction*

The isolation in a problem of at least two essential features is a necessary condition for the successful solution of direct and indirect problems in a single operation. Feature I is a feature by which the generalized statements of both direct and indirect problems are distinguished from one another in various arithmetical operations. The "symbolic" words *more*-*fewer*, *gave*-*took*, etc., appear in problems as variants of Feature I. The generalized statements of those direct and indirect problems that are not distinguished from one another by Feature I are distinguished from one another by Feature II. An example of Feature II is *Where is there more*? or *Who has more*? Generalized variants of Feature II were defined in Report IV in the following manner: In one variant the unknown is in the unsigned part of the problem's formula; in the second variant the unknown stands in the signed part of the formula. For example:

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*Translated by Harvey Edelberg.

1) $A + B = X$; 2) $A + X = S$.

In order to facilitate the future comparison of various methods of teaching the solution of problems, it is important to present the solution of the problem diagrammatically. By way of illustration, let us take the generalized statements of direct and indirect problems containing a single operation and the words *more* and *fewer*.

I. *Direct problems*:

1) There were A objects of the first kind. There were B more objects of the second kind than of the first kind. How many objects of the second kind were there?

2) There were A objects of the first kind. There were B fewer objects of the second kind than of the first kind. How many objects of the second kind were there?

II. *Indirect problems*:

3) There were A objects of the first kind. There were B more objects of the first kind than of the second kind. How many objects of the second kind were there?

4) There were A objects of the first kind. There were B fewer objects of the first kind than of the second kind. How many objects of the second kind were there?

The corresponding diagrams for solving these problems will be as follows:

I. *Direct problems*:

1) (I - a, II - a) ---- 0_a;

2) (I - b, II - a) ---- 0_s;

II. Indirect problems:

3) (I - a, II - b) ---- O_s;

4) (I - b, II - b) ---- O_a.

In all four diagrams the following notations have been used for the features that form the statements of the problems, and for the operations that must be performed in solving the problems: I - a, I - b are two variants of Feature I; II - a and II - b are two variants of Feature II; O_a denotes the operation of addition; O_s denotes the operation of subtraction. A comparison of the diagrams shows the following: a) direct problems 1 and 2 differ in various arithmetical operations only by Feature I; b) the same may be said about indirect problems 3 and 4; c) the direct-indirect problem-pairs--1 and 3, 2 and 4--are distinguished only by Feature II; d) finally, the direct-indirect problem pairs, 1-4 (requiring identical operations) and 2-3 (also requiring the identical operation) are simultaneously distinguished by both Features I and II.

According to regularity A, in the differentiation of complex objects, it is the different features of those objects that acquire predominant significance, that is, a person's reactions are determined chiefly by the different features rather than by the similarities. At the same time, the identical "features" become ineffective, that is, they do not themselves determine a person's reactions. In this connection, it is not only variants of the feature encountered earlier that become ineffective, but all other variants of the feature as well.

According to regularity A, in order to isolate a certain feature in problems and to form operations according to that feature, the differentiation must be such that the problems would be distinguished only by that feature. Isolation of Feature I requires the juxtaposition and differentiation (that is, the generation of different reactions) of direct problems by different operations (see diagrams 1 and 2 above), or of indirect problems by different operations (diagrams 3 and 4). Isolation of Feature II must occur if the direct and indirect problems denoted above (in diagrams No. 1 and 3, as well as in 2 and 4) are differentiated. In order to guarantee the effectiveness of both essential features of the problems, the differentiation of all of the pairs of problems listed here is apparently required.

We should note that not every solution of pairs of different problems is responsible for the advisability of using regularity A and of isolating necessary features. Here those problems requiring identical arithmetical operations--Numbers 1 and 4 and Numbers 2 and 3 (see diagrams)--may serve as an example. The preservation of the same operation in the problems will be a modification of unessential features only if the essential features of the program do not vary. The conversion of a problem with diagram 1 to a problem with diagram 4--as with the conversion of type 2 to type 3--does not mean a modification of unessential features, since, in this connection, Features I and II both change as well. Another example would be a pair of problems composed on the principle of variation of only the unessential features (an alteration of "plot" or numerical data).

Report VI:

A Comparison of the Effectiveness of Some Methods of Teaching the Solution of Indirect Problems*

The existing methods of teaching the solution of problems in the primary grades do not ensure the effectiveness of all the essential features of the problems. In Report V two features were pointed out whose isolation is indispensable for direct and indirect problems to be solved successfully in one operation.[14] The general deficiency of existing methods of instruction consists, in part, of the inexpedient utilization of regularity A in singling out the above mentioned features in problems.

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*G. M. Bakhromeeva and E. I. Galakhova of Moscow School No. 4 were coauthors of this report, along with B. B. Kossov. Translated by Harvey Edelberg.

[14] Feature I--This is a feature by which the generalized statements of both direct and indirect problems are distinguished from one another into various arithmetical operations. The "symbolic" words more-fewer, gave-took, etc., appear in problems as variants of Feature I. The generalized statements of those direct and indirect problems that are not distinguished from one another by Feature I are distinguished from one another by Feature II. An example of Feature II is "Where is there more?" or "Who has more?"

In fact, according to an arithmetic textbook [15] and the current curriculum, first-grade pupils solve only direct problems. But since direct problems can be distinguished only by Feature I (see Reports IV and V), the differentiation of direct problems into various operations can and does lead to the predominance of variants of Feature I. Feature II has only one variant in direct problems, and those problems are not distinguishable from one another by this feature. According to regularity A, the variant of Feature II in these circumstances is not supposed to determine the pupils' operations, and actually does not determine them. This non-signalness of Feature II in direct problems is also transferred to the feature's other variant, contained in indirect problems. For this reason, variants of Feature I also predominate in the indirect problems with which the pupils first come in contact; and it is no accident that pupils solve indirect problems the same way they solve direct ones.

In the second grade the pupils turn to indirect problems on finding the minuend and addend. Their problem book introduces these problems without sufficiently contrasting them with direct ones--or making any connection at all between the direct and the indirect--and there are sections of that book in which problems of only one type are selected. Such headings as "Problems on Finding the Unknown Minuend" and "Problems on Finding the Unknown Addend" are typical of these sections. Indirect problems differ among themselves just as direct problems do--in Feature I only. According to regularity A, the concentrated solution of indirect problems alone promotes the dominance of Feature I. Moreover, the very same variants of Feature I that call for a single arithmetical operation in direct problems, correspond to the opposite operation in indirect problems (addition instead of subtraction and vice versa--see the diagrams of problem solutions in Report V). In circumstances where Feature I is dominant, an alteration of connections occurs among the pupils between operations, on the one hand, and the variants of Feature I, on the other. After such an alteration the reintroduction of direct problems--and this is demonstrated by current instruction practices in the schools--is frequently accompanied by errors in their solution.

Devices usually employed by teachers and recommended by methodologists to make the solution of indirect problems easier for students amount either to a variation in the verbal formulations of Features I and II, or to the conversion of indirect problems into direct ones by the insertion of the corresponding symbolic words. A principle known in the literature as the variation of unessential features forms the basis for these devices, which may be useful at specific stages in instruction. By themselves, however, they cannot guarantee the effective isolation of all the essential feature of the problems or the development of the necessary associations. The authors of arithmetic textbooks follow precisely the logic of the variation of unessential features when, in the first-grade problem book, they offer only direct problems while, in the second-grade book, they list 14 problems in succession on determining the minuend and then 21 successive problems on finding the addend. In each of these cases only one variant of essentia Feature II is given. Instruction practices in the schools indicate that all of this contributes little to the isolation of all the essential features of problems and to the development of the ability to solve problems.

Finally, many authors point to the opposition itself of direct and indirect problems as one measure for overcoming mistakes when solving indirect problems. There is no doubt that this particular method has been underestimated up to now. It is usually discussed superficially, and then only after other methods; it is no accident that the method is ignored in textbooks and curricular guides on methods.

One may assume that the number of common features in the differentiate items is significant for the successful isolation of the essential, distinctive feature. It is possible that the necessary feature can be bet isolated under conditions in which a minimal number of other common featur exist, i.e., by removing all superfluous, distracting components and by "baring" the two essential problem-features mentioned above. In order to verify this assumption, an investigation was carried out in which the resu of two methods of teaching the solution of problems were to be compared: 1) the method of immediately differentiating complete statements of the problems, 2) the method of at first differentiating only some simplified models of the problems.

Report IV showed that, of the three types of indirect problems, those of Type 3 with the words more-fewer gave pupils the most trouble. At the same time, there were grounds to expect that, with appropriate instruction, problems of Type 3 would be entirely comprehensible to first-grade pupils. In both methods of teaching, therefore, we decided to begin instruction immediately with the more difficult indirect problems to Type 3. We also thought it possible that, with successful instruction, the ability to solve these more difficult problems would be a positive factor in the solution of the easier problems of Types 1 and 2.

Our instruction by the first method adhered closely to the methodology of Bantova, who achieved good results in the second grade (although she did use a sufficiently large number of exercises to solve the problems [3]). Unlike Bantova, however, we began instruction in the first grade, and therefore, limited ourselves to one variety of the Type 3, indirect problems--problems of increase and diminution. The number of exercises used in solving such problems (and therefore the total amount of time expended on instruction) was smaller in our case. We tried to create other, equivalent conditions of instruction in the two grades while preserving the fundamental differences between the methods of instruction. In this way we hoped to achieve comparable conditions for characterizing the two methods.

In the first method of instruction the teacher would fully explain the solution of a pair of problems such as the following (one direct and one indirect):

1) Six mushrooms were growing under a fir tree; three mushrooms more than the number under the fir tree were growing under a birch tree. How many mushrooms were growing under the birch tree?

2) Six mushrooms were growing under a fir tree; three mushrooms more than the number under the birch tree were growing under the fir tree. How many mushrooms were growing under the birch tree?

The solution of these problems is diagramed as follows:

1) $(I - a, II - a) \rightarrow 0_a$; 2) $(I - a, II - b) \rightarrow 0_s$.[15]

The second pair of problems (numbers 3 and 4) differed from the first pair in only one word: Fewer was substituted for more. Thus, the solution diagrams for the second pair are:

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[15] I - a, I - b are two variants of Feature I. II - a, II - b are variants of Feature II. 0_a denotes the operation of addition, 0_s denotes subtraction.

3) $(I - b, II - a) \longrightarrow 0_s$; 4) $(I - b, II - b) \longrightarrow 0_a$. The contrast between the problems in each pair, in accordance with regularity A, promoted the isolation of Feature II; while the contrast between problems 1 and 3 or problems 2 and 4 had to contribute to the isolation of Feature I. Indirect problems were solved by converting them into direct ones; for example, if $a > b$, then $b < a$, and, in accordance with the word *fewer* ($<$) subtraction was required. Moreover, the juxtaposition of a direct and a corresponding indirect problem should have protected the pupils from erroneously converting the direct problem. The teacher accompanied the verbal explanations and the problem texts themselves with appropriate drawings on the blackboard, but all of the fundamental relations between quantities were given, of course, in verbal form.

Such is the peculiarity of teaching indirect problems on the basis of their convertibility into direct ones. One typical characteristic of this first method of instruction is the minimal difference between problems with respect to any one feature and, at the same time, the presence of a large number of common constituents. Thus, problems 1 and 2 mentioned above fully coincide lexically (in nineteen common words) and differ only in the location of the words *birch tree* and *fir tree*, that is, they differ only in Feature II.

In the second method of instruction we tried to reduce as much as possible the number of common constituents in the problems. By continuing to simplify situations, we finally obtained certain models of elementary situations (problems) in which many features characteristics of direct and indirect problems were lost, but the two essential features--I and II-- were preserved. "Model" situations were represented by two pairs of assignments corresponding to the number of simulated problems.

The first assignment was a model of the first direct problem in addition that was discussed above: Each pupil had a set of colored mugs on his desk. Using this set, the pupil had to put n red mugs on the left side and n blue ones on the right side. Then the teacher asked that the number of blue mugs be made m greater than the number of red ones. The children added another m to their n blue mugs.

The second assignment served as a model of the second problem--an indirect one. Once again, the original n red mugs and n blue ones lay before the children and again they had to make the number of blue mugs m greater than the number of red ones, but this time not by changing the number of blue mugs. Instead, the students, with the teacher's help, removed m mugs from the quantity of n red ones. In this way one quantity grew larger than the other by addition to the first or subtraction from the second. Here we see Feature I--the word more, and Feature II-- the color of the mugs to which it is necessary to add (one variant of Feature II) or from which it is necessary to take away mugs (another variant of this feature). Thus, as befits the models, the general diagrams of the situations coincide with the diagrams of the solutions to the first (direct) and second (indirect) problems above.

In the third and fourth assignments the word more was replaced by the word fewer, and the pupils' work was organized accordingly. Thus, the model assignments--when compared with the problems--were distinguished from one another by a smaller number of words and did not require calculations and verbal conversion into another (more customary) form of assignment. Such was the first stage in the second method of instruction.

In the second stage the teacher employed texts of direct and indirect problems in juxtaposition--just as in the first method of instruction--but without using the principle of reciprocity. It was necessary to build a little connecting bridge from the first-stage assignment to the solution of the problems, since many of the children (about 40%) were unable to do this independently. For that purpose we taught the children to assume first that two numbers--a known and an unknown--were equal to each other (". . . first let us suppose that there are as many under the birch tree as there are under the fir tree. . ."). The subsequent course of work was identical to the one followed in completing the assignments in the first stage of instruction.

Experimental instruction by the first method was conducted in grade 1-A of Moscow School Number 4 at the end of the 1960-61 academic year; simultaneously, experimental instruction by the second method was being carried out in the parallel grade--1-B. Before this special instruction began, grade 1-B had no advantage whatsoever over grade 1-A in the level of arithmetic preparation. The number of pupils who correctly solved the descriptive, indirect problems of Type 3 did not exceed 9 to 11.5% of all the pupils in the class--a figure corresponding to the average shown by the first and second grades in a number of other schools that we investigated.

The results of instruction by the two methods showed the superiority of the second, model method of instruction (the one employed in grade 1-B). This was expressed by the following: 1) In grade 1-B 61.8% of the pupils were already solving the control assignments correctly when doing their first control work after a single, first-stage assignment. In grade 1-A the first control work was not conducted until the completion of three assignments, and even then the corresponding percentage was only 57.1. After seven assignments, however, the percentage reached 64.7--hardly towering over the index of 61.8% achieved in grade 1-B after just one assignment. 2) In all five of the control exercises conducted in the two grades after the same number of assignments, the results were better in grade 1-B. 3) With the same amount of time devoted to instruction in both grades (8 assignments of 20 minutes duration each), the highest achievement in grade 1-B (90.6% solved correctly) exceeded that for grade 1-A (71%). The reliability coefficient for the difference in results equals 2.0. 4) In grade 1-A symbolic errors in direct problems were encountered in all control exercises. The percentages of pupils who committed these errors in the five control exercises are as follows: 2.9; 15.6; 8.6; 26.5; 12.9. In grade 1-B, on the other hand, not a single such mistake was encountered during the same period .

One must assume that the reliability of the difference calculated above in point 3 actually would be higher if everything that was mentioned in all four points were considered. Moreover, the model instruction in grade 1-B could have been even more effective had we made the transition from the first to the second stage somewhat earlier. By the same token, it was evidently possible to reduce the total number of assignments in grade 1-B by roughly one-third without damaging the results.[16]

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[16] Several ways for increasing the effectiveness of model instruction will be treated in another report.

The merits of the model method of instruction became apparent in the solution of both direct and indirect problems which had been given only in verbal form (without visual support), and which contained two-digit numbers (rounded tens). In grade 1-B the percentage of correct answers in the first solution of such problems was 81.8.

During this instruction, therefore, facts were obtained which relate to an understanding of the general mechanisms of the operation of regularity A. The differentiation of two objects by one distinctive feature (the isolation of that feature) occurs faster when there is a smaller number of other common features.

Report VII:

The Degree of Abstraction of Learning Material and Its Role in the Formation of Generalized Associations*

In pedagogical practice there are often cases where pupils know the verbal formulations of rules well, but do not always act in accordance with them. The pupils' incorrect operations originate in incorrect associations. The first terms of such associations either exclude certain relevant features of objects and phenomena, or include irrelevant features (or both occur).

Let us call relevant those features which are contained in precise rules, and irrelevant those features which are not contained in the rules. Let us further agree that a feature with signalness is a feature on which one's operations depend in some way. The signalness of a certain feature has many causes. Specifically, as was shown in Report III, in the differentiation of two complex objects their distinctive features generate signalness, that is, one's reactions are determined by variants of the same features which are inherent in different objects. At the same time, features which are not used to distinguish objects from each other--identical features--become non-signaling, that is, do not themselves determine reactions. This regularity was designated for brevity, regularity A. If the differentiable objects being used as learning material are distinguished not only by relevant but also by irrelevant features, then the latter, because of regularity A, may have signalness and prompt incorrect operations in the pupils.

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*G. V. Usanova and E. M. Sharonova of Moscow School No. 22 coauthored this report with B. B. Kossov. Translated by Patricia A. Kolb.

Learning material can differ in its degree of abstractness (concreteness). An object's abstractness depends upon the number of features that are intrinsic to it as opposed to other objects--the fewer such features, the greater the abstractness of the object. For example, the letter addends in the expression a + b are more abstract than the addends expressed by the concrete numbers 5 + 3. The first expression, the letter one, cannot be altered on the basis of the comparative value of the addends; in a letter expression, we usually abstract ourselves from this feature. At the same time, this supplementary feature is typically present for a numerical expression (aside from all features held in common with the letter expression, for example, the presence of two different addends).

Let us suppose that we want to make a certain feature of objects signaling, using regularity A. For this purpose, let groups of objects of a variable degree of abstractness be used. Then the question arises of what degree of abstractness must be preferred. Theoretically, it is better to take more abstract objects. Since the number of their features is the most limited, it is easy to select those objects that will always be distinguished from each other by relevant features only. On the other hand, because of the negative manifestation of regularity A, irrelevant distinctive features can become signaling in more "concrete" objects. The goal of this investigation is to test the validity of the above-stated theoretical proposition on the possible advantages of using more abstract learning material in instruction in the first grade.

During the first semester of the school year 1961-62, in two first-grade classes of School Number 22 in Moscow, we instructed the children in the concepts of the whole and its parts.

The choice of these concepts, which were not included in the elementary school arithmetic curriculum, was determined by two considerations. First, as we shall see below, these concepts were convenient for an investigation of the role of abstract material in first-grade instruction. The degree of abstractness of the appropriate learning material could be varied easily. Second, these concepts were necessary for teaching first graders the general methods of solving direct and indirect problems.[17]

As an example, let us take direct and indirect problems in addition and subtraction in one operation, problems in finding a sum, difference, addend, minuend, and subtrahend. All of these problems can be written in the form of the following equations: 1) $A + B = ?$; 2) $S - A = ?$; 3) $? + B = S$; 4) $A + ? = S$; 5) $? - B = A$; 6) $S - ? = B$; where ? is the unknown, A and B are parts, and S is the whole.

Proceeding from the concepts of the whole and the part, it is possible to solve all of these equations without resorting to more complicated algebraic concepts. It is sufficient to operate in accord with one of two rules: In order to find a whole, the known parts must be added; in order to find a part, the other, known part must be subtracted from the known whole. Thus, it is necessary to learn the concepts of the whole and the part in order to master the general methods of solving problems by means of equations.

The general plan of instruction was as follows:

1) The initial period of forming the concepts of the whole and its parts with the use of concrete numerical quanities. The methodology of instruction in both classes was identical during the initial period.
2) The introduction of more abstract numerical quantities in one of the classes. In the other class, instruction proceeded as before.
3) General control work in the two classes in order to compare the mastery of the concepts of the whole and parts and the mastery of the necessary operations.

The initial period of instruction was divided into two stages. First, we demonstrated visually to the pupils that a whole object or a whole set (an apple, a group of children) could be divided into two parts; after rejoining (drawing together) these parts, we would again be able to see the whole. Second, in teaching the first ten numbers, we encouraged the pupils to consider each number (except the unit) as a whole composed of parts. As a result of the work, the pupils mastered the following: A whole is composed of parts; in order to obtain the whole, it is necessary to add up its parts; in order to obtain one part of the whole, it is necessary to subtract from the whole its other part.

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[17] An investigation by Kossov [6] showed that such general methods were within the grasp of first graders.

The concepts developed thus were used for notating examples in addition and subtraction: The pupils were supposed to put two dots under the whole and one dot under each part. Initially, the children learned to place the dots correctly in examples with concrete numerical quantities only. The corresponding assignments were of two types; they can be illustrated by two columns of examples:

I.	II.
3 + 2 = 5	3 + 2 = 5
2 + 3 =____	____ = 5
5 - 3 =____	____ = 2
5 - 2 =____	____ = 3

In completing each of these two assignments it was initially necessary to copy the first example in a given column, to fill in the dots (symbolizing the whole and the parts), and then to solve the rest of the examples in the column, constantly keeping in mind what the whole and the parts were in the first example. The teachers gave the following instructions for assignment I: "Fill in the answers for the rest of the examples. In order to find the answer more easily, look at the first example; the whole and the parts there are the same as in all the other examples." Assigning problem II, the teacher usually said: "Fill in the empty spaces to complete the examples. In the second example, think of how the whole 5 can be obtained from the same parts (that is, from 3 and 2) as in the first example. In the third and fourth examples, think of how one part can be obtained from the whole and the other part which you see in the first example."

Generally, the pupils placed the dots correctly and solved examples within the limits of the numbers 1-10 (they had not yet begun the second ten).

On the sixth day of the experimental instruction, we introduced a column of two-digit examples in both classes: 63 + 29 = 92; 92 - 63 =   ; 92 - 29 =   ; 29 + 63 =   . (On the previous day the children had learned to distinguish two-digit numbers, to find equals among them when written on the board, and to copy two-digit numbers from the board into their notebooks.) In both classes, the isolation of the whole and parts in two-digit examples was completed with inadequate but approximately *equal* success: Somewhat more than 50% of the pupils in each class placed the dots correctly in just one of the four examples. Thus, despite knowledge of the verbal rules and correct application of these rules within the limits of the first ten numbers, many pupils had not yet mastered the generalized method of finding the whole and the parts. Apparently, instead of sufficiently broad generalized associations (for a detailed definition of this concept, see [20]), most of the pupils developed narrow associations; moreover, the first terms of these associations included some irrelevant features inherent in operations with numbers within the limits of 1-10.

Further instruction in one of the classes (I-A) was modified for greater abstractness in the quantities used in the arithmetical operations. Two notations were used for the operations: ? + ? = ?; ? - ? = ?. Thus, we made the operation sign the essential, distinctive sign of these notations--all of the other components were identical. When almost all (80%) of the pupils in class I-A had learned to isolate the whole and parts in such abstract notations correctly, we conducted control work in both classes. Let us cite one version of the problems (all work was usually conducted in two variants):

10 + 2 =
2 + 10 =
12 - 2 =
12 - 10 =

a + b = c
c - b =

The result was that 81% of the pupils in class I-a but only 41% of the pupils in class I-B completed all of the problems correctly. The difference in the results was statistically significant: The difference exceeded its standard deviation 2.6 times. It was interesting that the pupils in class I-A who made mistakes in placing the dots were primarily those who had not yet mastered this operation in the abstract examples. Consequently, the indicated difference in the results of the two classes will increase because of the *positive* influence of class I-A's training in abstract examples. If one considers the number of mistakes made in placing the dots, then the advantage of class I-A becomes distinct. As is evident from the table, the pupils in class I-B made a certain number of mistakes in finding the whole and the parts, even in examples with concrete numerical quantities. The nature of the errors in these examples indicates that the pupils in class I-B made associations that

Table 6

Isolation of the Whole and Parts

Results / Class	Number of placements of dots			
	in letter examples		in concrete examples	
	correct	incorrect	correct	incorrect
I-A (Abstract Instruction)	49 (94%)	3 (6%)	104 (100%)	0 (0%)
I-B (Regular Instruction)	40 (69%)	18 (31%)	109 (93%)	8 (7%)

were too narrow. The most typical mistake was the use of the number 10 as the whole, although it was not a whole in a single example in the control column (see above). For example, $\underset{\cdot}{12} - \underset{\cdot}{2} = \underset{\cdot\cdot}{10}$. The origin of the error becomes apparent when it is recalled that operations within the limits of the numbers 1-20 had been introduced only two days before the control work. Previously, the children had solved examples only within the limits of the first ten numbers, and the number 10 had actually always been the whole. The incorrect, narrow association (10 as the whole) causes mistakes in the transition to the next range of numbers (the second ten). The nature of the errors made by class I-B in the control work and in succeeding days indicates that, in some cases, this erroneous association was somewhat more generalized: In general, whichever of two numbers was the larger was used as the whole. For example: $\underset{\cdot\cdot}{16} + \underset{\cdot}{3} = \underset{\cdot}{?}$ Moreover, sometimes both associations cited here, the narrow and the more generalized, occurred simultaneously. For example, $\underset{\cdot}{2} + \underset{\cdot\cdot}{10} = \underset{\cdot\cdot}{12}$ (lest one of the large numbers be slighted!).

All of these facts provide grounds for supposing that for many first graders, operations with numbers within the limits 1-10 are determined not by the generalized rules for finding the whole and its parts, but rather by incorrect associations involved with irrelevant arithmetical examples. Such associations did not occur when abstract "examples" of the type (? + ? = ?) were used in class I-A.

Thus, abstract examples helped the pupils in class I-A to isolate the essential features in all examples and contributed to the development of sufficiently generalized associations according to the general rules for finding a whole and a part. Consequently, the use of learning material of a different degree of abstractness has significant value for the development of generalized associations. It is important to note that the pupils in class I-A also solved an adequate number of concrete problems. It must be assumed that the exclusive use of very abstract material, without sufficient use of the concrete, would not have positive results. In fact, this abstract material (letters, question marks), taken by itself, becomes just as concrete as numbers. Thus, the exclusive use of such material would, in all probability, lead to the development of narrow and, moreover, completely useless associations.

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THE FORMATION OF GENERALIZED OPERATIONS AS A METHOD FOR PREPARING PUPILS TO SOLVE GEOMETRY PROBLEMS INDEPENDENTLY

E. I. Mashbits*

The pupils' independent solution of new problems, as is known, is preceded by a state of instruction that demands the pupils' active participation. Pedagogical guidance of the pupils' activity should be based on the psychological regularities of forming a problem-solving method. Elucidation of these regularities is an important problem for psychology, both theoretically and practically.

In order to outline precisely the problems subject to investigation here, let us present the solution of a mathematical problem as a system of operations[1] $S_1 \dashrightarrow S_2 \dashrightarrow \ldots S_n \dashrightarrow R$, where S_1 are the given conditions and R the desired result. From the logical standpoint, the solution of any problem is already contained in its condition, and the essence of the solution consists in transforming the information contained in the condition with the help of supplementary information--logical rules of this transformation. Such an approach can be explained by the fact that logic does not consider the (implicit or explicit) form in which the condition is given, whereas from the standpoint of psychology the translation from implicit to explicit form means that the person solving the problem acquires new information. It is clear from this that, for the subject, the supplementary information required for the solution of a problem consists not only of logical rules for transforming the geometric material, that is, rules for establishing connections between the elements of the chain $S \dashrightarrow R$, but also information contained in each element of this chain.

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[1] Here the symbol $\dashrightarrow$ does not mean implication, but is used to indicate any kind of connection.

Thus, the algorithm for solving problems (for example, geometry problems) contains operations that differ essentially. These are primarily operations (we call them operations of mathematical logic) with which the solver transforms each element of the chain into the following one. These operations are logical transformations of mathematical material; they are abstracted from the subject content of the concepts and may be applied to different concepts regardless of their content. The second type of operation (mathematical) consists of operations performed within each element of the chain S -→ R. They are defined as the relationships between concepts included in an operation and are always dependent on subject matter.

The pupils' generalized mastery of the system of mathematical operations composing the algorithm of the solution of a problem is a necessary but insufficient prerequisite for its solution. Before the mathematical operations can be applied in the correct sequence, it is necessary that the pupils have mastered the operations of mathematical logic. Therefore, if we express the process of instruction in solving problems of a particular type in terms of a program, we must distinguish in it two subprograms. The first one should contain the system of strictly mathematical operations that enter into the algorithm of solving problems of the given type, and the second should contain the system of operations of mathematical logic that enter into this algorithm.

A subprogram is a system of operations (they are sometimes called "information blocks" or simply "blocks") each of which in turn consists of definite levels differing in their method of expressing the study material. In principle there are two possible approaches to presenting the study material (in our investigation, mathematical operations and operations of mathematical logic) in one and the same "information block." The first is with the use of "logical models," that is, models that reveal the structure of an operation by reducing it to a set of other, elementary operations. Such models are called logical models insofar as they are used in computers where a "complex" operation is broken down into a set of elementary ones. As applied to man, we define elementary operations as those operations already formed and whose performance evokes no difficulties. The logical model may be presented as a formalized system of elementary operations whose ordered implementation leads to completion of the model operation.

A second approach is to apply "psychological models" that model the relationship between concepts at various levels of generalization. To create such models one must study the process of operation formation in the pupils. Because this process had not yet been sufficiently investigated, we conducted a special experiment.

In this communication we present data relating to the first subprogram, that is, to the formation of mathematical operations.

The investigation was conducted using material on pupils' solution of right triangles. This material was chosen because: 1) its mastery by pupils evokes significant difficulties, 2) in mathematical structure these problems differ from problems solved earlier, 3) the necessary operations have not been formed in the pupils before studying this topic, 4) problems of this type make possible a precise accounting of the knowledge and operations required for their solution as well as the composition of a practically useful algorithm.

The first experiment consisted of two series. The aim of the first series was to study the process of the formation in pupils of mathematical operations under teaching conditions. We formulated the operations needed for mastering the concepts of the trigonometric functions of an acute angle, finding the size of an angle from its trigonometric function, and finding the magnitude of the trigonometric function of an acute angle from the size of the angle. We also formulated the operations for establishing the relationship between the trigonometric functions of supplementary angles.

The aim of the second experiment was to ascertain how pupils apply the operations learned in the first series and the information related to them and how this formation of mathematical operations occurs when the pupils are solving problems independently. The pupils have to master the operations necessary for establishing the relationships between the trigonometric functions and the sides of a right triangle and for making the transition from one trigonometric function of an acute angle to another function of this angle or its supplement.

If in performing these operations a pupil experienced difficulty, he was given an auxiliary problem representing a model of the given operation. The modeling of mathematical operations was the fundamental methodological device of the experiment. Further, each operation was presented to the pupil as an independent action. That is, the means for attaining the goal (the solution of some problem) became the content of the goal. In other words, the auxiliary problem was required to disclose the relationship between the concepts contained in the structure of an operation. But the relationships themselves in different models of a single operation were presented differently. Presented below are models of the operation of establishing the relationships between the trigonometric functions of an acute angle and the sides of a right triangle.

Isolating the operation as an independent action. The first model is the notation of the trigonometric function of an acute angle (for example, $\sin \alpha = \frac{a}{c}$) with the instructions "determine one side using the trigonometric function and the other side."

In the second model the unknown quantity is denoted by x, $\sin \alpha = \frac{x}{c}$ (in the experiment it was established that it is easier for the pupils to perform an operation when the unknown is expressed by x or y); the requirement of the problem remained as before.

In the third model the given formula was presented as the ratio $d = \frac{a}{c}$ (thus all its elements were named); it was required to find the unknown member of the ratio.

The fourth model presented the formula as an arithmetical operation of division; the requirement was stated as finding the dividend (divisor) through the quotient and the divisor (dividend).[2]

The first experiment was conducted with 27 eighth-grade pupils. All problems had to be solved aloud. The subjects' argumentations were noted in detail or recorded on tape.

The results of the investigation permitted us to isolate four stages in the formation of the strictly mathematical operations in the algorithm of the solution of the problem.

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[2] In formulating the relationship between the trigonometric functions of an acute angle and the sides of a right triangle, the pupils were allowed to use a drawing.

I. The pupils do not separate the essential from the nonessential features of the concepts included in an operation. They are unable to isolate the relationships between the essential concepts. When performing an operation, pupils rely on visual and spatial schemes. When one alters these schemes (position of the drawing, its notation) for the forms of expressing the concepts in the operation, the pupils are faced with significant difficulties that impede their correct completion of the operation. Even simple operations such as finding the size of an angle from its trigonometric function and finding the trigonometric function of an acute angle from its size are not reversible; each is recognized independently of the other.

II. The pupils grasp the relationships between essential concepts of an operation but cannot generalize these relationships. Therefore, the form of expressing concepts still influences the success of completing the operations, especially when the alteration of this form is connected with a higher level of generalization (for example, from $d = \frac{a}{c}$ to $\sin\alpha = \frac{a}{c}$) or when altering the form of expression of the concepts leads to significant change in the structure of the operation, such as omission of some particular element (for example, the replacement of numerical data by letters in the operation, or the transition from one trigonometric function of an acute angle to another function of this same angle or its supplement). The operations are reversible only within a particular level of generalization of the relationships between concepts (for example, the pupils can find a and d in the equation $d = \frac{a}{c}$, but when d is replaced by $\sin\alpha$ they cannot do this; they correctly note that $\sin 37^\circ = \cos 53^\circ$ and $\cos 53^\circ = \sin 37^\circ$, but cannot determine that $\sin(90^\circ - \alpha) = \cos\alpha$.

III. The relationships between the concepts are generalized by the pupils, and the operations are carried out correctly no matter how the concepts are expressed. The operations are reversible, and the pupils are aware of each of their directions as a part of the operational structure. But the structure itself is regarded narrowly--as two directions (forward and reverse) of a specific operation.

The pupils still cannot establish the connection between the different operations that constitute the broader operational structure (for example, $f(\alpha) \longleftrightarrow \alpha \longleftrightarrow \beta \longleftrightarrow f_1(\beta)$).[3]

IV. This stage is characterized not only by the generalization of the relationships between concepts, but also by the pupils' establishing the connections between the various operations. Because of this the operational structure is formed in them as a system of operations. In addition, the pupils master the reversibility of not only an individaul operation within the operational structure (for example, $f(\alpha) \longleftrightarrow \alpha$ or $\beta \longleftrightarrow \beta$), but also a system of operations (for example, $f(\alpha) \longleftrightarrow f_1(\beta)$).

Insofar as each successive stage of forming operations has, relative to the preceding one, a higher level of generalization of the relationships between concepts, it may be viewed as a specific level of the operation. Such levels, as our investigation showed, give a general picture of the operation formation process, but the presence of each of them is not a requisite. The real process of forming operations depends largely on the pupils' individual peculiarities of their mental activity.

The data obtained through the investigation permitted us to classify subjects into three groups, depending on the types of difficulties they experience in the formation of operations and how these difficulties in mastering and generalizing the operations were overcome. Let us examine the process of forming mathematical operations (in the conditions of the special instruction and when the pupils were independently solving problems) in subjects of the different groups.

It was characteristic of the subjects of the first group (fourteen pupils) that they could not single out the essential features of the concepts contained in the structure of an operation. Hence complete mastery of even relatively uncomplicated operations such as finding the leg of a right triangle (either adjacent to or opposite an acute angle) required painstaking work. The subjects of this group, when performing a series of operations, at first based their thinking on visual spatial schemes. Their inability to generalize the relationships between concepts made it necessary to give the pupils various models, gradually generalizing the relationships between concepts and varying the form in which they were presented. Special models were required that showed both directions of an operation so as to make it reversible. Each direction of the operation was first mastered by the pupils separately; only later were the operations mastered as a part of the operational structure.

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[3] α and β are acute angles of a right triangle; $f(\alpha)$ and $f_1(\beta)$ are different trigonometric functions of these acute angles; $\longleftrightarrow$ is the sign for equivalence of the relationships.

One may judge how the pupils mastered operations in solving problems by the fact that only two of them mastered the relationship between the trigonometric functions of an acute angle and the sides of a right triangle after being shown the second model ($\sin \alpha = \frac{x}{c}$), nine pupils needed to be shown the next model also, and three had to see model 4.

In subjects of the second group (eight pupils) the process of forming operations proceeded similarly. The pupils often confused essential features of concepts with nonessential ones and did not consider the whole system of essential features of the concepts that enter the structure of an operation, but only some of them. Hence, they made mistakes. The pupils' mental operations were not flexible or generalized enough. This resulted in the fact that, for example, when mastering the relationship between the trigonometric functions of supplementary angles, the pupils, finding equal relationships in the trigonometric functions of different angles, could not independently conclude the correlation between the trigonometric functions of supplementary angles. Unlike the first group, the members of the second group needed less assistance.

In the third group of subjects (five pupils) generalization of the relationships between concepts was more successful than in the others. Their mental operations were generalized and dynamic. Thus, it was easy for them to think even when the form of expression of the concepts was altered. The subjects also had no trouble making the transition from direct relationships between concepts to the reverse relationships, that is, the operations were developed in two directions, forming a definite operational structure. The subjects of this group were able to master the mathematical operations while solving problems with a minimum of assistance. They easily combined the individual operations into a system.

The data obtained show that in forming mathematical operations, these operations must be isolated into an independent operation. In other words, the operation should first appear not as a means to an end (the solution of some problem), but as the content of the goal of the operation. That is, it should be set apart as an independent problem. Formation of generalized mathematical operations is promoted by modeling the relationships between concepts, that is, by applying psychological models. Systems of such models should provide for variation of the concepts and the forms of expressing them, isolation of their individual links in a complex operation, and the transition not only from expanded to abbreviated operations, but, conversely, from abbreviated to expanded operations.

It was noted above that besides psychological models one may also utilize logical ones. Logical models reveal the structure of the modeled operation by reducing it to elementary operation. In our investigation we obtained some information on the relative effectiveness of the logical and the psychological models, but since the criterion of the effectiveness of the models should be extended beyond the framework of one subprogram, these data will be presented along with a description of the characteristics of the formation of operations of mathematical logic that are contained in the algorithm of solution.

AN EXPERIMENTAL INVESTIGATION OF PROBLEM SOLVING AND MODELING THE THOUGHT PROCESSES*

D. N. Zavalishina and V. N. Pushkin

As indicated by Glushkov [3], the external approach to modeling man's mental activity is characteristic of the contemporary stage of the development of cybernetics. Basic to it, as is well known, is the behaviorist scheme, which views each operation as the probable result of a stimulus, and the processes resulting in the operation are ignored as taking place in a "black box."

Yet representatives of cybernetics already take into account the one-sidedness of this viewpoint and understand that, for example, to create computing apparata capable of forming algorithms unforeseen by the curriculum, it is necessary to reveal the mechanisms of man's mental activity [2]. The regularities of thought important for cybernetics can be ascertained through experimental investigation of a person's solution of problems. At this the methodology should make possible quantitative analysis of data and be appropriate for programming and putting into a machine, for transmission to a machine, the devices and methods of human thought that are discovered during psychological investigation. The methodology will include, on the one hand, verification of the results of psychological study of thought, and, on the other, its practical outlet.

The methodology applied in our work consists in the following. On a blank with 6 squares (designated a, b, c, d, e, f), five numbered slips of paper are randomly placed so that one square remains empty (Figure 1). The problem is to put the slips into normal order, i. e., 1, 2, 3, 4, 5 (with square d empty) by making a series of "simple rook's moves," which means moving some slip of paper onto the adjacent empty square with each move (Figure 2).

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a 3	b 2	c 5
D	e 4	f 1

Figure 1

a 1	b 2	c 3
D	e 4	f 5

Figure 2

This problem is a variant of the mathematical "game of 15" [1]. It is known that certain mathematical propositions and concepts have been applied to this game, especially those of higher algebra (permutation, inversion, etc.), which allow one to calculate all possible situations and ascertain solvable and unsolvable situations.

By definition, permutations of n elements are combinations of them that differ only in the order of the constituent elements; hence each situation in our problem can be viewed as a permutation. The total number of such permutations, that is, variants of situations, can be expressed as $P = n! = 5! = 1\cdot 2\cdot 3\cdot 4 = 120$. But not all of these permutations are solvable. To be solvable, a permutation must lead to the abovementioned normal position (normal permutation). The solvability of a given permutation is determined by 1) the evenness of the normal permutation and 2) the place position of the empty square, or the "imaginary slip" with the number 6. There are even and noneven permutations. Whether a permutation is even or noneven is determined by the number of inversions in it.

Inversion is the term for the mutual positions of two slips in which the slip with the larger number stands before the slip with the smaller number. Into the total number of inversions in a permutation there enter the inversions of the numbers of all skips making up the given permutation, with the number 6 which is assigned to the "imaginary slip" in the empty square (square d in this case).

In our case, the normal permutation is even. Since, for convenience, we decided the initial situation with d the empty square, then all even permutations leading to the normal form through an even number of movements of the slips are solvable. In our case there are $\frac{1}{2}\cdot 5! = 60$ such permutations.

With the aid of this methodology the experiment was conducted as follows. The subject was presented with one of the situations and given these instructions:

> Move the slips around in normal sequence (1, 2, 3, 4, 5), using the free square and moving the slips vertically or horizontally only. You cannot change the slips by switching two of them. You also cannot move them diagonally. In the final situation square d should be empty, as it is in the initial situation.

The subject's every move was recorded. For simplicity of fixing the moves during the experiment, each move is denoted by the number of the slip moves. For example, solution of the initial situation of $\begin{smallmatrix}253\\41\end{smallmatrix}$ was done by the subject through this sequence of moves of the numbered slips:

4 5 3 1 5 4 2 3 1 5 4 2 3 1 2 3 1 2 3 4 5 3 2 1 (24 moves).

On the basis of this notation, if we know the initial situation, we can reproduce the course of solution with the aid of notation similar to that of chess. The first move is slip 4 from square e to square d (4ed); 2: 5be; 3: 3cb; 4: 1fc; 5: 5ef; 6: 4de; 7: 2ad; 8: 3ba; 9: 1cb; 10: 5fc; 11: 4ef; 12: 2de; 13: 3ad; 14: 1ba; 15: 2eb; 16: 3de; 17: 1ad; 18: 2ba; 19: 3eb; 20: 4fe; 21: 5cf; 22: 3bc; 23: 2ab; 24: 1da.

This real course of solution is related to the optimal one, which consists of 16 moves: 1: 4ed; 2: 5be; 3: 2ab; 4: 4da; 5: 5ed; 6: 1fe; 7: 3cf; 8: 2bc; 9: 1eb; 10: 5de; 11: 4ad; 12: 1ba; 13: 2cb; 14: 3fc; 15: 5ef; 16: 4de.

One of the merits of this methodological device is the possibility of regulating the complexity of the problem. Thus, for example, the situations $\begin{smallmatrix}243\\15\end{smallmatrix}$ and $\begin{smallmatrix}413\\25\end{smallmatrix}$ can be solved in 4 moves; $\begin{smallmatrix}235\\14\end{smallmatrix}$ and $\begin{smallmatrix}135\\42\end{smallmatrix}$ can be solved in 6 moves; $\begin{smallmatrix}415\\32\end{smallmatrix}$ and $\begin{smallmatrix}124\\53\end{smallmatrix}$ in 8 moves: $\begin{smallmatrix}325\\41\end{smallmatrix}$ in 14 moves; $\begin{smallmatrix}253\\41\end{smallmatrix}$ in 16 moves. Despite the complexity of the problem being determined by several factors (interconnection of elements, etc.), there exists a definite correlation between the complexity of the problem and the number of moves of the optimal variant of the solution.

In creating a machine program that solves this problem, one may take two approaches. By the first method the optimal solutions of all 60 situations are registered in the machine, which needs then only to recognize the situation and to produce the appropriate variant of the solution. This method is the least interesting from the standpoint of cybernetics. A much more interesting method of principle interest is a program set up such that the machine knows only the initial and final situations and the method of moving the slips. Hence the machine works from the same instructions given to the subjects. Here a knowledge of those intellectual operations with whose help a man solves this problem becomes significant.

The present series of experiments (107 experiments on 14 subjects) permitted us to ascertain the meaning of one of those operations--the activity of establishing connections between the elements of the problem situation. It was observed that various forms of the solution of a problem are in a direct relationship to the expressed connections between the elements. An analysis of the records makes it possible to state these three forms of the solution of a problem.

The first form is characterized by a course of solution based on the expression of the individual elements outside their mutual connection.

> Record of the experiment. Subject V. A.: situation: $\frac{235}{14}$.
>
> Solution process. Moves 1 and 3, then 2 and 1. Begins moving the slips at random: 3 2 5 4 2 5. "It's not coming." Again moves slips: 1 3 5 2 4 1 3 5 2 4 1 3 4 2 5 4 2. "It's more or less clear that 4 and 5 are in place." Moves: 5 4 2 5 1 3 5. "I don't get it. Let's try it this way." Moves: 1 3 5 1 2 4 3 2. "1 and 5 have to change places, then everything would be in order." Moves: 1 5 2 3 4 1 5 2 3 5 2 3 5 4. "Finally! I didn't have to make the moves, just think up a plan first."

The problem was solved after 56 moves. The optimal variant is 6 moves: 1 4 5 3 2 1. Because the subject did not reveal the connections between the elements, the movement of the slips was random and chaotic.

The special characteristic of the *second form* of solution is the transfer of slips on the basis of seeing the connections between several elements of the situation.

Record of the experiment. Subject V. A.; situation: 415
32.

> *Solution process*. Looks at the situation. "I don't see a plan yet." Separates all slips into two groups, 1, 2, 3, and 4, 5. Continues analysis: "1 and 3 can be set up correctly, but 5 makes for confusion. 5 is between 1 and 3. 1, 2, 3 are in order. We have to put 5 on square *e*." Moves: 4 1 5 2 3 5 2 3 5 4.

The problem was solved in 10 moves. The optimal variant is 8 moves: 4 1 3 2 5 3 2 4. As can be seen from the record, there is no whole plan of solution; all elements of the situation are divided into *two* groups, and the slips are moved on the basis of the correlation of the elements within these groups. Hence, movement of the slips is no longer random.

The following record can also serve as an example of how the expression of the interaction of elements determines the course of solution.

Record of the experiment. Subject V. P.; situation: 451
23.

> *Solution process*. "I don't see a plan. Maybe if we move 4 down then 1 goes to its own place. But in planning this transfer we observe that in this case 4 and 5 end up reversed. How can we make it so both groups 1, 2, 3 and 4 and 5 are placed correctly simultaneously? I'll begin with 2, because 5 has to be chased into the corner; then, I think, it'll be easier--but I still don't see it." He moves: 2 5 1 3. "I see the solution of the problem, the plan is ready." He realizes the plan" 5 2 4 1 2 4. The problem is solved optimally.

The *third form* consists in seeing at one time the entire solution of the problem from beginning to end. In this case there is present the representation of *all* elements of a situation in their interconnections and relationships.

The investigations show that the basis of the algorithm for the machine solving problems of this type should consist of the modeling of the process of establishing connections among the elements of the problem situation. We can assume that just this establishment of connections eliminates the need for a large number of operations on sorting out all the variants, which is characteristic of contemporary problem-solving computer mechanisms.

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THE COMPOSITION OF PUPIL'S GEOMETRY SKILLS

A. K. Artemov*

Report I: Making Auxiliary Geometric Constructions**

Success in solving many geometry problems is often attained by correctly making auxiliary constructions. In actual school work one may often observe pupils making these constructions by the "trial and error" method. Finally a trial proves successful and the problem is solved. The pupils sometimes become convinced that their attempts are fruitless and give up trying to solve a problem. Thus, it would seem that success in choosing auxiliary constructions is completely accidental.

In the methodological literature there are various statements concerning the methodology of teaching children how to make auxiliary constructions. Nemytov [13] considers it difficult to point out any definite rules in solving such questions; what is needed is imagination and creativity on the part of the pupils. He does not, however, explain what he means by these two concepts. Other methodologists try to give some rules (advice) that might help in figuring out what auxiliary constructions are needed. Hadamard [7] recommends determining the "givens" by conventional notions. Danilova [6] suggests, for example, continuing straight lines until they intersect, forming triangles, etc. However, these recommendations cover only some cases encountered in solving problems, and they remain theoretically unsubstantiated. On the whole, the methodology of instruction in making auxiliary constructions remains undeveloped, and the pupils possess no special devices. It is suggested that solving a large number of problems _per se_ will lead to the formation of a high level of abilities in making such constructions.

In this work an attempt is made to examine two questions:

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1. What is the basis of the children's guesses as to the choice of expedient auxiliary constructions?
2. Is any kind of special instruction in making auxiliary constructions possible?

Ascertainment experiment

The intent of the experiment was to ascertain students' basis for guesses concerning the selection of auxiliary constructions. The subjects had a relatively high level of ability in solving geometry problems. The subjects were 17 third-year students of the Physics and Mathematics Department of the Penza Pedagogical Institute. They had average and above-average abilities in mathematics, and in their second year and partly in their third year, had taken a special course in elementary geometry, in which they had solved many diverse problems. The experimental material consisted of problems from the school workbook by Rybkin [17] designed for the sixth and seventh grades (problems 16, 18 from section 6, and others). The experiment was conducted like ordinary auditorium examinations. In solving problems at the blackboard, the subjects were asked to reason aloud. The answers were recorded and later analyzed.

During their previous instruction in geometry, the subjects had learned no special methods for seeking expedient auxiliary constructions.

Results of the experiment

1. Many subjects made "blind," random, auxiliary constructions. Often such constructions were inexpedient and did not facilitate solving a problem. It follows that solving even a large number of problems is not sufficient for forming a high level of ability to make expedient auxiliary constructions.

2. In individual subjects the selection of auxiliary constructions was based on several general operational propositions. Operational propositions are those in which there are indications of what must be done in the concrete situation for solving a problem.

Here are some examples. It was required to prove that (under given conditions) one angle was three times as large as another. Subject L. began solving the problem in this way. "Here we must compare two angles. They must be taken together." Then he fulfilled this statement in a drawing by making auxiliary constructions. The problem was solved correctly.

It was required to prove that (under given conditions) one chord in a circle was larger than another. Subject E. dropped perpendiculars from the center of the circle to the two chords. When the experimenter asked why that was done, the subject answered, "To make a comparison. We know that the larger chord is closer to the center." Consequently the subject was acting on the basis of a general operational statement--to compare two chords, one must compare their distances from the center of the circle.

These operational propositions correspond to a generalized association of this type: recognition of the initial data of the assignment, then recognition of another (secondary) assignment by means of which the given problem is solved. Since making such an association enabled the subjects to make expedient auxiliary constructions and to solve the given problem, it follows that the existence of a large store of such associations and their actualization is one of the necessary conditions for successful mastery of skills in making auxiliary constructions.

Let us agree to say that guiding associations are those associations underlying the selection of an expedient auxiliary construction that leads to the correct solution of a problem.

3. The experiments showed that the subjects command a very poor store of operational propositions and the guiding associations connected with them. This is the essential obstacle in mastering problem-solving skills.

Analysis of the correct solutions of problems

The analysis consisted of an attempt to increase the number of guiding associations revealed in the ascertaining experiment. The essence of the analysis consisted in establishing the pattern of reasoning that leads to the auxiliary construction realized as the problem is being solved correctly. The material consisted of the written work of the tenth graders in School Number 4 of Penza, together with the solutions of problems given in Barybin's collection [3] . This collection was selected because it contains solutions of problems on the entire topic of the school geometry course.

Let us cite a concrete example of such an analysis.

Problem [3: No. 98.]
In parallelogram ABCD (see Figure 1): $BC = 2AB$; M is the midpoint of AD; E is the base of the perpendicular dropped from point C to the extension of side AB. Prove that $\angle DME = 3\angle AEM$.

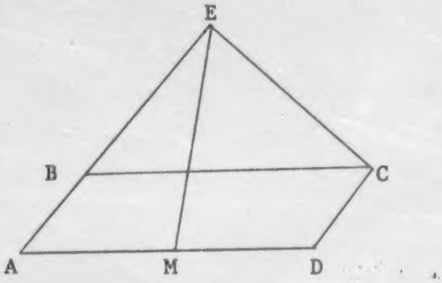

Figure 1

Barybin made these constructions: $MM_1 \perp EC$, and point M is connected to point C. (See Figure 2.)

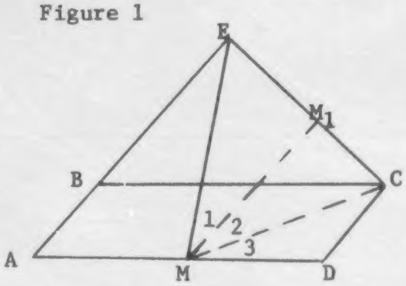

FIGURE 2

Let us dwell on the first one. In the proof it is shown that $\angle 1 = \angle 2 = \angle 3 = \angle 4$. Consequently, the supplementary construction is based on these operational propositions:

a) to prove that $\angle DME = 3\angle AEM$, we must divide $\angle DME$ into three equal parts, each of which would equal $\angle 4$;

b) to construct $\angle 1 = \angle 4$ within $\angle DME$, we must draw through point M the line $MM_1 \parallel AE$ (or $MM_1 \perp EC$).

Generalized associations having the abovementioned standard features correspond to these operational propositions.

Of course, with such an analysis one can establish only the possible operational propositions that lead to the given constructions. It is quite possible for the problem to be solved on the basis of the actualization of some other associations, such as the ones underlying the blind attempts. This, however, is unimportant for us, since the aim of the analysis is to look for operational propositions (and their guiding associations) that can lead to a guess at the selection of an auxiliary construction.

In analyzing correct solutions we were able to establish three types of guiding associations:

1. Recognition of the peculiarities of objects as given in the conditions of the problem leads to recognition of other properties of these objects.

For example, the recognition that a segment and an angle are given, and under the angle this segment is visible, evokes the notion of an arc of a segment containing the given angle.

2. Recognition of the terms entering into the conditions of the problem leads to recognition of the definitions of the concepts they denote.

For example, if the term "an angle between two planes" is given in the conditions, the corresponding construction is made on the basis of the definition of the concept signified by this term. This type of association often corresponds to the recommended rule of replacing the concepts with their definitions [6,7].

3. Recognition of the originally given assignment leads to recognition of another (secondary) assignment whose solution will lead to completion of the first.

Within this type we were able to establish eight kinds of generalized guiding associations. Here are examples of several of them.[1]

a) Recognition of the assignment to find (calculate) the size of some segment (angle) leads to recognition of the assignment to construct this segment (angle) on the drawing in connection with the given elements of the drawing so as to obtain an auxiliary figure from which one might compute the unknown object.

For example, recognition of the assignment to calculate the altitude of a trapezoid, given a lateral side and the angle of the slope of this side to the base, leads to recognition of a secondary assignment of constructing this altitude on the drawing from the end of the given side (the vertex of the upper base of the trapezoid, and not at some other point) with the aim of forming a triangle from which the unknown altitude may be calculated.

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[1] Compilation of a detailed list of guiding associations was not the goal of the present article.

(b) Recognition of the assignment to compare two quantities leads to recognition of the assignment to find and compare other quantities such that by reasoning correctly on the basis of their comparison, the unknown relationship can be established.

For example, recognition of the assignment to compare the lengths of two oblique lines leads to recognition of the task of comparing the lengths of their projections and to construction of the projections (if the latter are equal, on the basis of a known theorem one may conclude that the oblique lines are equal).

Trial teaching experiments

The aim of these experiments was to see how the abilities to make auxiliary constructions are perfected under the influence of exercises promoting the indicated types of associations. The subjects were tenth graders from School Number 4 of Penza. First, all were given the assignment to solve this problem in writing:

> The diagonals of an isosceles trapezoid are mutually perpendicular; prove that the midline of the trapezoid is equal to its altitude. [15:Problem 456].

Of those pupils who were unable to solve this problem, 9 with average or below-average mathematics ability were selected. Four class sessions were held with them during which problems as difficult as the one above were solved. During the solution the pupils' attention was drawn to the fact that in making auxiliary constructions they must orient themselves to special general (operational) propositions (corresponding to those associations cited above). These operational propositions the pupils wrote down in their notebooks as rules that they then used in solving problems, especially when making auxiliary constructions. Then the pupils solved a control problem approximately as difficult as the first written assignment. Of the 9 pupils, 7 made the auxiliary construction correctly and solved the control problem; only 2 made useless auxiliary constructions.

These experiments allow us to conclude that a guess of expedient supplementary constructions is based on the actualization of generalized associations. It may be supposed that the formation in pupils of generalized guiding associations of the types indicated definitely has an influence on increasing the level of the abilities to make auxiliary constructions for solving geometry problems. Nevertheless, further supplemental experiments are necessary to ascertain the most expedient ways of forming these guiding associations.

Report II: The Cause of Errors Connected with the Concept of the Plane*

In the present work we are presented with the problem of determining the causes of very widespread errors connected with the concept of the plane arising in the teaching of solid geometry in secondary school. The errors under consideration have not previously been the subject of a special study.

Description of an Error

The following problem was posed to ninth grade students[2] in their fifth lesson on solid geometry (the beginning of their study of it) to be solved orally (The sketch was done by the teacher at the blackboard.):

The straight line MN and the plane P of the parallelogram ABCD have two common points, M and N (Figure 3). How is the point F situated in relation to the plane of the parallelogram?

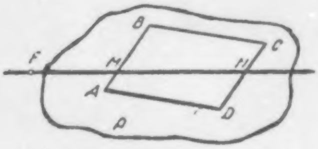

Figure 3

In the solution, it was determined that four of the students did not regard point F as belonging to plane P. Several of them said that the point F could have been regarded as belonging to the plane P if that point had lain in the interior region bounded by the wavy line; others asserted that the point would belong to the plane of the parallelogram if it were inside that parallelogram. When the teacher asked why they thought so, they answered, "It's obvious from the drawing." However, the correct answer is that the point F, under the given conditions, always belongs to the plane P.

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*Translated by Nancy Stetten.

[1] School Number 4 in Penza (school year 1960-61).

Supposed causes of the error

From a comparison of the above situations, in one of which the students gave the correct answer and in the other incorrect ones, the following suppositions about the reasons for the errors can be made.

1) The concept of the "plane" is identified with the concept of the "part of a plane bounded by some enclosed figure."

2) The pupils adequately understand the term "plane," but they understand the term "the plane of a given figure" as that part of the plane contained within that figure.

3) The pupils' words, "It's obvious from the drawing," provide grounds for supposing that the students may have solved the problem by actualizing direct associations: recognition that the point is inside (outside) an area with respect to an outline leads to recognition that the point belongs (does not belong) to the plane P.

Analysis of these suppositions (starting with the third). A direct association is, so to speak, an association with a "short circuit." Awareness of a rule (or definition, theorem, or whatever) with which to carry out an operation is not part of its makeup. As Shevarev [19] established, direct associations are formed as a result of repeated performance of exercises of a single type and are actualized when the pupils recognize the situation facing them as familiar and well-known. In the instance under consideration, such exercises had not been completed in the solid-geometry lessons. Thus there are grounds for supposing that the pupils solved the problem by actualizing direct associations formed in the study of solid geometry.

However, it is possible to assume that such an association had been formed at some previous time, for instance, in the study of plane geometry, and actualized in solving the present problem. Actually, in the course in plane geometry, one must sometimes solve problems whose drawings show some object located outside a given figure or in its internal region--such as a point inside a circle or outside it.

Thus, we must suppose that in the study of plane geometry, there have been conditions for forming the indicated association. However, the probability that this association was actualized in the condition of solving the given problem is insignificant, because the situation in which the pupils had to solve the problem was completely new to them; it was a new branch of geometry, with new concepts and new definitions. The terms of the problem differ substantially from the terms of problems in plane geometry, where the question of whether or not a point belongs to a plane is not asked.

An individual experiment was conducted in order to clarify further the reasons why the four students made this mistake. They were given an analogous problem, where the drawing represented a plane figure other than a parallelogram. The experiment was conducted six days after the lesson at which the previous problem had been solved. During this time, there were no geometry lessons. Two pupils repeated their error. From conversation with them, it emerged that one of them identified the concept of the plane with the concept of the part of the plane enclosed by the wavy line (Figure 3). For him the point F belonged to the plane only if it was located within the outlined area. If the point F was located on the wavy line, he considered the location of the point to be "on the end of the plane P." The other pupil was adequately aware of the term "plane," but understood the term "plane of a parallelogram" as the part of the plane located within the outline of the parallelogram. For him, the point F belonged to the plane of the parallelogram only if it was located "inside" that parallelogram. Situating the point F in the interior region of the wavy outline but outside the parallelogram ABCD meant that the point F belonged to the plane P, but did not belong to the plane of the parallelogram (Figure 3).

In order to exclude the influence of chance circumstances in determining the reasons for the mistake, one more experiment was conducted. The subjects were 16 pupils in tenth grade.[3] The experiment was conducted in the Lesson 32 of solid geometry. Each subject was given a small card with questions to which they had to give written answers. There was no opportunity to cheat.

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[3] School Number 7, Penza (school year 1962-63).

The way three pupils answered the question "What is a plane and how can it be represented?" indicated that they identified the concept of the "plane" with "the part of the plane bounded by an outline." For example, subject Ch. wrote: "A plane is a surface bound by a closed curve." Pupils also wrote that the plane could be represented in the form of a rectangle, a closed curve, a circumference, the top of a table, etc.

Five pupils gave answers that suggest the conclusion they did not adequately understand the term "plane of a given figure." For example, subject Sh. responded to the question above: "The plane is infinite. We can bound it and get a square, a triangle, and other figures. These bounded parts of the plane will form the plane of the square, the triangle, etc." The foregoing provides a basis for the following conclusions.

1) The error under consideration was consistent and was exhibited by various students learning from various teachers in various schools. Consequently, the reason for the error must arise from general features of instruction.

2) The experiments confirmed the above-stated supposition on the reason for the error we are studying, namely, that the error was caused by the actualization of one of two erroneous associations: a) consciousness of the term "plane" ⟷ consciousness of a certain part of the plane bounded by an enclosed figure as adequate content for the concept of the plane; b) consciousness of the term "plane of a given figure" ⟷ consciousness of the part of the plane enclosed within a given figure.

3) The reason for either error can vary for different students. How is it possible that the above-mentioned erroneous associations are formed?

From the very beginning of the study of geometry in the sixth grade, the concept of the plane is formed on the basis of the pupils' concepts of objects in the real world. These objects, naturally, are not of unlimited dimension. Thus, Kiselev's textbook [8], which our subjects were using, says that the surface of a good windowpane or the surface of still water in a pond will provide a notion of the plane. The plane is described in approximately the same way in the new textbook by Nikitin [14]. The team "surface" is essential in these descriptions of the plane. However, some pupils understand this term to mean only the shape in which part of the surface is bounded. Later, in the study of solid geometry, the teacher, of course, explains that the plane should be understood as continued infinitely in all directions. However, the content of the textbook in solid geometry by Kiselev [9] again orients the student to an awareness of the plane as the internal region surrounded by an outline. This is obvious, if only from the fact that the plane as a geometric object is compared with objects whose surfaces have a rectangular shape. This rectangle is depicted in the drawings in the form of a parallelogram (with the exception of four figures in which the plane is outlined by an arbitrary line). An essential property of the plane-- its unboundedness-- is used very rarely, and then in contradiction to the illustrations presented.

For example, parallel planes are defined as planes that do not intersect no matter how far they are extended. However, theorems following this sign for parallelism of planes, as well as other theorems, are proved in conformity with little pieces of planes, depicted in the form of parallelograms intersecting within the limits of the drawings. This also promotes the formation of the erroneous associations noted above. Apparently, we must work out a special system of exercises directed at an adequate understanding of the term "plane."

The actualization of the erroneous associations indicated above results in other pupil errors that are well known in practical school instruction:

1) If the vertex of a pyramid is projected orthogonally onto the plane of the base at a point lying outside the base of the pyramid, some pupils believe that the point does not belong to the plane of the base.
2) Students in the ninth grade[4] were given the problem: Through the midpoint of two lateral edges and the center of the base of a regular triangular pyramid draw a plane and determine the shape of the resulting section. Many pupils constructed the

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[4] School Number 49, Penza.

section as shown in Figure 4.

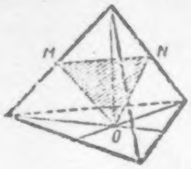

Figure 4

If we proceeded from the assumption that these students understood the term "plane" as a part of a plane enclosed within a certain figure, which, as we have seen, is quite plausible, then the error we have examined is easily explained: The term "to draw a plane," in this case, is understood as "to connect given points with straight lines," i.e., to construct a bounded figure.

3) In problem Number 13 of Section 11 of Rybkin's book of problems [18], where one is asked to construct a section passing through the axis and a lateral edge of a regular triangular truncated pyramid, some tenth graders[5] made a representation as shown in Figure 5. One must suppose that here the term "plane" is inadequately realized, as in the previous instance.

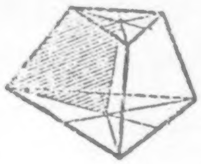

Figure 5

The low level of pupils' spatial concepts has been advanced in the literature as a reason for such errors as these. However, this is an extremely general explanation. As we have seen, the matter is considerably more complicated.

Report III. Solving Geometric Problems by Using Drawings*

The selection of instructional methods for the formation of schoolchildren's skills in solving geometric problems is closely related to the answer to the question whether the process of solving these problems is or is not reduced to the actualization of previously mastered knowledge. This thesis is almost obvious. Really, if the process of solution as a whole is based only on the actualization of previous knowledge, then what has been studied earlier must be repeated in every possible way. If there is something else in this process, then along with the repetition, different techniques of instruction should be devised.

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[5] School Number 4, Penza.

*Translated by Joan W. Teller.

In the methodology of teaching mathematics, the question just formulated is not posed or solved directly. There are only individual general observations, relating to this question in some measure. Thus, Bradis [4] writes that school problems are usually solved on the basis of certain statements from the theoretical course, and the chief difficulty consists in the proper selection and combination of these statements. Consequently, one can make the assumption that Bradis does not reduce the process of solving a problem, as a whole, to the actualization of former knowledge.

Chichigin [5] believes that the difficulty in the pupils' search for a solution to a geometric problem consists in selecting that theorem or formula, from the ones previously studied, which will lead to the solution. Consequently, one can make the assumption that, according to Chichigin, the process of solution is reduced only to the actualization of previous knowledge.

In a psychology textbook [16] it is noted that solving a new problem consists in establishing new ties (associations) among the knowledge previously acquired. On the grounds of this general conclusion, one might believe that new associations arise and later are used in solving geometry problems. But what are the peculiarities of the newly-arisen associations? This question remains unsolved, both in mathematics methodology and in psychology. Nevertheless, an answer to it has a great significance for the practice of instruction. The present work poses the problem of revealing the peculiarities of some associations that arise in solving geometry problems. We shall have in mind only those problems that are different in content but are solved on the basis of using the same scheme, which is familiar to the pupils. Problems of this sort are very often encountered in the study of geometry.

The method of the investigation used an analysis of the process of correct solution to a geometry problem. Each operation (addressing attention to something--the pronunciation, of a definite word or sentence aloud or to oneself, and the like), realizable in solving the problem, was examined as the second part of some assocaition. Then, necessary and sufficient conditions appeared that provide for the flow of the processes composing the second part of the association. These conditions were taken as the first part of the same association. This method was used by Shevarev [19] in an analysis of the processes of a correct solution to algebraic problems.

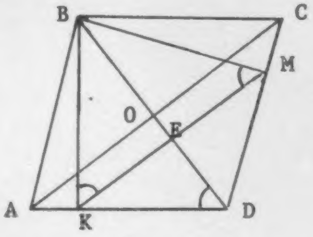

Figure 6.

Problem [2: No. 533]. From the vertex of an obtuse angle of a rhombus (see Figure 6), perpendiculars are dropped to its sides. The length of each perpendicular is equal to a, and the distance between their bases is equal to b. Determine the area of the rhombus.

This problem is of average difficulty; its solution, which was carried out by the authors, is based on the use of a scheme well known to the pupils (the base of the rhombus is multiplied by its altitude).

Solution. 1) According to the condition, BK=BM = a and KM = b. Therefore,

$$KE = \frac{b}{2} \quad \text{and} \quad BE = \sqrt{a^2 - (\tfrac{b}{2})^2}.$$

2) $\triangle$ BKD is a right triangle. Therefore $BK^2 = BD \cdot BE$ (by the theorem on proportional lines in a right triangle). Then

$$BD = \frac{a^2}{\sqrt{a^2 - (\frac{b}{2})^2}}.$$

3) The isosceles triangles ABD and BKM are similar ($\angle BKM = \angle BDA$ as angles with mutually perpendicular sides). Then AD:BD = BK:KM. Hence,

$$AD : \frac{a^2}{\sqrt{a^2 - (\frac{b}{2})^2}} = a : b; \quad AD = \frac{2a^3}{b\sqrt{4a^2 - b^2}}.$$

4) The area of the rhombus is $S = \dfrac{2a^4}{b\sqrt{4a^2 - b^2}}$.

(BK is the altitude of the rhombus.)

The conclusion established in the second step are clearly based on the following assertions:

a) △ BKM is isosceles;

b) BE is the median of the triangle BKM;

c) △ BKE is a right triangle, BK is the hypotenuse of the triangle BKE, KE and BE are the legs, and so forth.

Not one of these assertions is given in the problem's conditions; they are new.

For an objectively correct answer to the problem's question (What is the area of the rhombus with the given conditions?), △ BKM should always be recognized as isoceles in the solving process. This was essential, for example, in the third step of the solution. In other words, each time the problem is solved, the perception or visual notion of the triangle BKM should be followed by an awareness of the quality corresponding to it. In the absence of this awareness, "advancement" of the solution becomes impossible. But this means that at the step under consideration a new association is formed: awareness of the triangle BKM ↔ awareness that it is isosceles.

At this step some more new associations arose, for example, awareness of the triangle BKE ↔ awareness that it is a right triangle, and others. Analogous phenomena are observed at other steps in the solution where new associations are also formed.

The following objection is possible: the associations mentioned are not new; they were formed earlier and were actualized in solving the given problem. Such an objection, however, is groundless.

First, in the theoretical sections of the school course, mathematical statements corresponding to the associations under consideration are not especially studied. For example, nowhere is the assertion studied that a triangle formed by a segment of a diagonal of a rhombus, a segment of a line parallel to another of its diagonals, and by a perpendicular dropped from the vertex of an oblique angle to the side of the rhombus is a right triangle (△ BKE).

Second, geometry problems, in contrast to algebraic examples, are not, as a rule, uniform. Therefore the probability of a repeated "encounter" with the triangle BKE in exactly the same position as is portrayed in the drawing, even if a problem with an analogous drawing had been solved formerly, would have to be admitted to be quite insignificant.

Third, the associations we studied do not arise at once in the solving of a problem, but are the result of certain previous processes.[6] For example a pupil says: "BK = BM. But we know that if two sides of a triangle are equal, it is called isosceles. Consequently, $\triangle$ BKM is isosceles." If we assume that the associations under consideration were already there, then the flow of these processes would be unnecessary. All of this speaks for the fact that these associations are actually newly arisen.

The indicated associations usually function only in that period of time when a given problem is being solved. After this, as a rule, the necessity of especially reinforcing the new associations that have arisen here does not appear. In solving other problems, other associations arise and function. If we even assume that a new problem is solved with the use of a sketch, similar within the limits of sight-estimation to the figure that was cited, then this in no way means that in the sketch for a new problem, for example, the diagonal BD of a rhombus should be recognized as the hypotenuse of triangle BKD. In the conditions of a different problem the same elements of a sketch would be recognized in a different way.

Associations that function only over a certain period of time are called periodic [19]. This term stresses their short-term existence. In the given case the term is only for the period of solving the problem. Consequently, the associations mentioned above are periodic.

Two classes of periodic associations that arise in solving geometry problems can be singled out easily. The formation of the first class is determined by the condition of the problem and by the designations adopted in the sketches. For example, it is given that the segment KM = b. This connection lasts during the whole process of solution and then is destroyed. We shall call these periodic associations specified. They are essential for obtaining an answer to the problem's question, but they do not advance the solution. In contrast to them, one can say that the advancement of the solution is based on associations of the second class, examples of which were cited above. Thus, for awareness of the similarity of triangles ABD and BKM, an awareness that they are isosceles is essential. Such periodic associations can be called advancement associations.

The above mentioned examples of periodic advancement associations have the same typical features. Awareness of the elements of a geometric figure is awareness of their conceptual character. We designate this type of association as Type A.

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[6] A description of these processes is not included in the task of the present work.

Associations of Type A are isolated. That is, the connection of the mental processes forming the association is related only to a definite element of a sketch. Moreover, every time these associations are actualized in solving a given problem, their first and second parts remain unchanged, since the sketch for a problem in the solving process does not undergo changes. Consequently, associations of Type A are constant.

Besides Type A, in the second class of periodic associations one can single out associations of another type, Type B. For example, at the third step in solving the problem, consider the moment preceding an awareness of the proportion AD:BD = BM:KM. For this proportion to be established, an awareness of the proportionality of the similar sides of the similar triangles ABD and KBM is necessary and sufficient. Consequently, at this moment a new association has also arisen: awareness of the similarity of triangles ABD and KBM ↔ awareness of the proportionality of their similar sides. Analogously, at the second step of the solution the association was formed: awareness that $\triangle$BKD is a right triangle and KE $\perp$ BD ↔ awareness of the quality, $BK^2 = BD \cdot BE$.

Both these associations, according to the considerations set forth above, are periodic, isolated, and belong to the same type, namely: awareness of a definite peculiarity of an object ↔ awareness of some condition in which a peculiarity of this sort is the subject. In our case such conditions will be the proportionality of the similar sides of similar triangles and the relation $BK^2 = BD \cdot BE$ in the right triangles. As is evident from the examples cited, the first parts of the Type B association are periodic Type A associations and the second parts are constant associations corresponding to the assertions studied in theoretical geometry courses and applied to specifically isolated elements of a sketch. Also, up to now we have examined only the rise of new associations. But in problem solving, the student's available "old" associations are actualized along with the formation of new associations.

The process of solving the problem from the very beginning was subordinated to one purpose-- to find the length of the side AD of the rhombus. This singleness of purpose is determined by the problem's question: What is the area of the rhombus? Undoubtedly, awareness of this question entailed actualization of the concept corresponding to the theorem, "The area of a rhombus is equal to the product of its side and its altitude." This association was actualized at the very end of the solution, when the value of the area was calculated. Operations stated in the solution were performed in full conformity with the named theorem.

Associations corresponding to general geometric statements studied in the theoretical course we agree to call ready-made associations. By using this term we wish to stress peculiarity of these associations-- they are usually formed under school conditions prior to solving the problem. Undoubtedly, under the conditions of instruction some advancement associations are transformed into ready-made ones.

Our analysis shows that the process of solving geometry problems with the use of sketches is quite complex. Both the actualization of previously learned associations and the formation of new ones occur. The latter circumstance, as we have seen, is not taken into account at all by the methodology of teaching mathematics. Nevertheless, success in solving geometry problems is possible only on the basis of skill forming new associations.

Report IV: The Occurrence of a Certain Psychological Phenomenon in the Solution of Geometry Problems*

In his analysis of the processes of solving algebra problems, Shevarev [19] established the existence of the following phenomenon. Suppose that in a particular segment of time a pupil deals only with complex stimuli, which can be represented on paper as A_1X, A_1Y, A_1Z, etc., where the letters A_1, X, Y and Z designate components of complex stimuli. Suppose that the student performs one response operation R_x when confronted with the first stimulus, another operation R_y when confronted with the second, operation R_z on the third, etc. (with the distinguishing characteristics of each response operation being determined by components X, Y, Z, . . . of the complex stimuli, the operations being identical in type). Then, in the absence of counteracting conditions, the distinguishing characteristics of component A (i.e., what distinguishes A_1 from A_2, A_3, etc.)[7] can become invalent (i.e., response operations R_x, R_y, and R_z appear to be independent of them). Only the components AX, AY, AZ, etc., will turn out to be valent. That is, when the pupil is confronted by the complex stimulus A_2X, for instance, he will perform operation R_x; when confronted by the stimulus A_3Y he will perform operation R_y, etc. In other words, the student will develop the generalized association $A_a \rightarrow R_a$, where a designates any one of the components X, Y, or Z, and the letter A designates the generic characteristics of A_1, A_2, and A_3. If A_1, A_2, A_3, etc., are among the nonessential components, then this is a valid association. But if these components are essential, that is, if when confronted by stimulus A_2X one should perform not operation R_x but another operation P_x, then such an association is of course erroneous. This error does not become evident, however, so long as the pupil is dealing only with complex stimuli of which A_1 is a part. It becomes evident only when he is confronted by a stimulus containing A_2 or A_3, etc. Conditions which counteract the development of an erroneous association are: a) awareness that A_1 has a necessary relationship to operations R_x, R_y, R_z, . . .; b) instruction by the teacher, after which the pupil becomes attentive to A; or c) realization of the error of operation R_x when performed in response to stimulus A_2X (if such instances have occurred).

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[7] The letters A_2 and A_3 designate components which are of the same type as A_1 but distinguished from it. Characteristics common to components A_1, A_2 and A_3 (generic characteristics) will be designated by the letter A.

*Translated by Ann Bigelow.

B. B. Kossov [10] found that this phenomenon also occurs in the study of arithmetic in the elementary school. The task of the present article is to determine whether this phenomenon occurs in the study of geometry on the secondary school level. It is very important to know whether such a phenomenon exists because this knowledge would allow us to understand the cause of certain widespread mistakes pupils make and to determine better methods for the study of mathematical (and in particular, geometric) material.

The investigation consisted of two parts. The task of the pre-experimental analysis was to reveal places in the high school geometry course that looked "suspicious" with regard to the appearance of the phenomenon being examined. Two types of material served as the basis for this analysis: 1) the author's observations of pupils' operations as he taught them mathematics, and 2) the contents of the standard high school geometry workbooks [15, 18].

We assumed that erroneous associations of the type described above could occur 1) when pupils are solving problems of the same kind one after another, where components of one type occur over and over, or 2) when the pupils know beforehand what "type" of problems they are going to be solving. The analysis of the contents of the standard workbooks showed that the selection of problems in them frequently meets these two conditions.

Let us examine the topic on "The Surface of a Pyramid," for instance. After studying the theorem concerning regular pyramids, students are usually given drill problems. The students know before they solve the problems that the content of the theorem about regular pyramids is what will be drilled. In the standard workbook by Rybkin [18] only problems on regular pyramids are presented with this theorem (Number 10) at first. These problems are subsumed under the general heading "Regular Pyramids."

Therefore, if the phenomenon formulated by Shevarev occurs in the study of geometry, we should find that the valence of the term "regular" is reduced. This means that as the children solve problems on regular pyramids, some of them may develop and drill associations between other components of the complex stimuli and the corresponding responses. For instance, they may drill the associations between the common characteristics of the numerical data in the texts of the problems and operations leading to the determination of a lateral surface of a pyramid according to the formula for regular pyramids. It can be expected that when the pupils are given problems on irregular pyramids, some of them will actualize these associations and thus make errors. For instance, on the diagram for such a problem they will perform operations inappropriate for irregular pyramids, calculate the lateral surface with the formula for a regular pyramid, and so on.

The experimental part of the investigation consisted of three series of experiments, of which the first two were investigative and the third was instructional. The experiments were organized in this way in order 1) to check whether the assumptions spelled out in the pre-experimental analysis were really justified, 2) to clarify which components of complex stimuli become invalent in specific instances, and 3) to find ways of preventing the negative influence of the phenomenon under study.

First series. The subjects were the pupils of two tenth-grade classes.[8]

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[8] At School Number 49 in Penza.

The experiment was carried out as a part of regular classroom activity.

Immediately following their study of the theorem of the surface of a complete regular pyramid, the pupils spent three class periods solving problems [18: Theorem 10] on calculating the surfaces of pyramids of this type alone. Such a sequence in the selection of problems for the experiment corresponds to their order in Rybkin's workbook. Only eight problems were solved. We considered the textual conditions of the problems and the diagrams corresponding to them as complex stimuli. The verbal components that these complexes share are the words "regular pyramid" in the texts of the problems. In this situation, without a clear awareness that a pyramid is given and not some other geometric figure it is impossible to make objectively valid responses. If, for instance, a rectangular prism were given, it would be necessary to perform other operations. The common visual components (i.e., those shown in the diagrams for all the problems) are: a) the "passing" of the altitude of the lateral faces through the center of the side of the base, and b) the "passing" of the altitude of the pyramid through the point of intersection of the medians or diagonals of the base in triangular and quadrangular pyramids, respectively. The response operation consists of multiplying the perimeter of the base times half the apothem.

The control problem was No. 16 from [18: Theorem 10]. It called for the calculation of the surface of a pyramid whose base is a parallelogram. To solve it correctly it is first necessary to find the area of two adjacent lateral faces and to double the result; in the diagram the altitude of a lateral face should not pass through the center of the side of the base.

In solving this problem 27 out of 46 pupils extended the altitude through the center of the side of the base, and nine tried to calculate the lateral surface using the formula for regular pyramids. Thus for these pupils the awareness of the word "regular" and the awareness of the position of the apothem of a pyramid had become invalent as they solved the preceding problems.

We must assume, therefore, that the formation of the erroneous generalized association shown above came as a result of the selection of problems.

Second series. The subjects were eighth graders.[9] Problems from a section on "Inscribed Angles" [15] served as the experimental material. In the seventh grade the subjects had solved most of the problems in this section using diagrams. The visual components these have in common are: a) the vertex of the angle is situated on the circumference, b) the sides of the angle form chords, i.e., the sides are situated within the circle, and c) the "ends" of the sides of the angle lie on the circumference.

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[9] At School Number 4 in Penza.

All the problems in this section are of the same type. We could assume, therefore, that solving them would lead to the formation of an erroneous association in which there would be no valent awareness of even one of the common characteristics. This assumption was tested in the following way.

Initially the content of the theorem of an inscribed angle (the measure of this angle is half the measure of the arc on which it rests) was repeated in class, and three problems [15: Number 659, 662, and 663] were solved by the class as a whole. Then the subjects were divided into two groups of approximately the same academic achievement. The members of each of the two groups were instructed to perform one variant of an assignment consisting of two problems.[10]

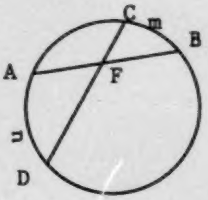

Figure 7.

Problem 660 from [15], the second problem of the first variant:

AB and CD are chords (Figure 7), $\overset{\frown}{CmB} = 40°$, $\overset{\frown}{AnD} = 60°$. Calculate $\angle$ AFD and $\angle$CFB.

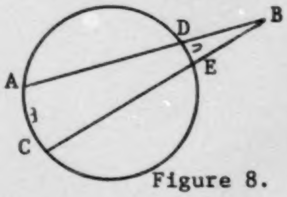

Figure 8.

Problem 661 from [15], the second problem of the second variant:

AB and BC are secants (Figure 8), $\overset{\frown}{AmC} = 60°$, $\overset{\frown}{DnE} = 30°$. Calculate $\angle$ ABC.

The first problems of both variants were the same as the preceding ones in content, while the second ones were control problems. But in the control problem of the first variant we modified only one of the common visual components of the preceding problems (we put the vertex F inside the circle), while in the second variant we changed the common components more radically, placing the vertex of the angle outside the circle and replacing the chords with secants. Our task was to find out how such a variation of the visual components might influence the occurrence of erroneous operations in the pupils.

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[10] The diagrams for these two problems were prepared by the experimenter and put on the blackboard, and the subjects copied them into their notebooks.

Of the thirteen pupils who performed the first variant of the assignment, seven made mistakes in solving the control problem. They said that ∠AFD = 30° but that ∠CFB = 20° (Figure 7). That is, they performed the operation that is correct for inscribed angles. In doing this they were not disturbed by the difference they found in the value of the two angles, which in fact are equal as verticals.

Thus when pupils solved the problems on inscribed angles in the sequence called for in the standard workshop, part of the visual components (for example, the position of the vertex of the angle on the circumference) became invalent for some pupils. What became valent was the aggregate of conditions consisting of a) an awareness of characteristics common to inscribed angles as well as to angles formed by intersecting chords, and b) the particular characteristics of the numerical data. The subjects' reactions were determined by this aggregate of data alone. In the course of the drill exercise on the theorem about inscribed angles, the pupils developed an erroneous association connecting particular visual aspects of the diagrams and the numerical data of the problems with the division of the arcs of the circumference into two. This association was erroneous from the very beginning. But its fallaciousness went unnoticed so long as they were solving problems on inscribed angles; it became apparent only when they began to work problems of another type.[11]

Fourteen subjects performed the second variant of the assignment. Seven subjects were unable to solve the control problems, but at the same time they did not make the mistake made on the control problem of the preceding variant, with the exception of one girl who said that ∠ABC = 30° (i.e., she performed the operation which is correct for the calculation of inscribed angles). Apparently these pupils also (or some of them) had developed the erroneous association described above, but they did not act on it.

Such results give us reason to believe that these subjects realized that the control problem was of a new kind rather than the same. Inasmuch as the "newness" consisted of a greater modification of visual components than the control problem of the first variant had, we can assume that the modification of several common visual components can hinder the actualization of an erronous association formed when many problems of the same type are solved one after another. This assumption needs to be checked further, however.

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[11] It is not relevant to the problem being discussed in the present work to give a detailed description of the erroneous associations that occur.

The task of the *third series* of experiments was to compare various procedures for selecting exercises to lower the valence for the awareness of separate components of complex stimuli. The subjects were eighth graders (also at School Number 4). Two theorems to which the pupils were being introduced for the first time were used as experimental material: 1) the perimeters of similar polygons are in the same relation to one another as any two corresponding sides, and 2) the areas of similar polygons are in the same relation to one another as the squares of the corresponding sides.

Drill on the content of the theorems consisted of solving problems. For the first theorem six problems were selected, concerning the calculation of the perimeters or sides *only of similar* polygons. To solve these problems accurately it was necessary to establish corresponding proportions. The pupil solved three of the problems with the teacher's help and three on their own (in a two-hour geometry class). The problems the pupils were to solve independently were given in two variants differing only in their numerical data. We assumed that with such a selection of problems the awareness of the term "similar" in the text and of identical visual components (a similarity in the shape of the figures) would become invalent for some of the pupils while the awareness of the three numbers making up the proportion given in the texts would become fully valent. The final problems (the seventh ones) were control problems. One of them had the following content.

> The base of an isosceles triangle is 28 cm and its perimeter is 126 cm ; the perimeter of a square is 252 cm ; find the length of one side of the square.

Before they began the pupils were cautioned that all they needed to do was answer the question, regardless of whether it was connected with the content of all of the conditions or only some of them.

If the assumption explained above is accurate, then in solving the control problem some of the pupils should have made up a proportion from the numbers given. Out of the forty eighth graders who took part in the experiment, twenty-three did solve the control problem by making up a proportion -- an incorrect operation. Of course, the pupils all knew perfectly well that the triangle and the square they had drawn were not similar figures. We must conclude, therefore, that the pupils' awareness of the similarity or difference in form of the figures drawn, as well as of the term "similar," had become invalent while they were solving the preceding problems.

Following the study of the second theorem problems were presented in the following sequence: the first, third, and fifth were on similar figures (concerning their areas), while the second, fourth, and sixth were from sections on geometry that had no connection with similarity. The first and second problems were solved by everyone together, and the rest by the pupils on their own (in two variants). Control problems on the content were similar to the one cited above.

Out of thirty-five pupils, six solved the final problem by making up a proportion, while the others produced the correct solution. Inasmuch as the experiments were carried out within a span of seven days, in the course of which there were no geometry classes, and the mistakes in the preceding work were not cleared up (and therefore the pupils' knowledge and skills did not change substantially), we must presume that the great reduction in the number of errors in the solution of the control problem in the second case as compared with the first was achieved by achieved by alternating the content of the preceding problems.

A repetition of the third series of experiments, using the same material and the same methods, was carried out on six other eighth-grade classes in various schools. As in the preceding case, the best results in the development of skills were obtained when the exercises were arranged so as to alternate. The overall results of the third series of experiments are given in Table 1. The column labelled "Number of Errors" indicates solutions of the control problem in which pupils made up a proportion.

Calculation of the criterion of reliability of difference in the two methods of instruction shows that its value is 4.4, considerably more than the critical value 2.6. This means that instruction averages significantly better results when the problems are arranged so as to alternate.

TABLE 1

RESULTS FROM THIRD SERIES

OF EXPERIMENTS

Arrangement of Problems	Number of Subjects	Number of Errors	Percentage of Wrong Solutions
with alternation	194	31	16
without alternation	200	70	35

The following conclusions can be drawn from all that has been said above:

1. The phenomenon of thought described above applies to the process by which high school pupils learn geometry.
2. One of the possible ways of anticipating and minimizing the negative influence of this phenomenon in the study of geometry is by alternating the content of the problems used for drill.
3. In day-to-day teaching it is important to foresee components of complex stimuli that could become invalent in the study of one aspect of geometry or another, in order to take steps to anticipate mistakes pupils might make. Accordingly, an appropriate analysis of the content of textbooks and workbooks, as well as a better arrangement of problems in each section of the curriculum, is urgently needed to improve methods of teaching mathematics.

Report V. Peculiarities of the Mastery of Mathematics in Similar Situations*

The study of many questions in secondary-school mathematics is implemented in situations for which the school material to be mastered is characterized by the following:

1. This material is identical in several features and distinct in only one feature;
2. Objectively correct response operations by the pupils are

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*Translated by Joan W. Teller.

determined by a clear awareness either of identical or of distinct features.

We shall call situations similar if they possess these traits.

It follows from the definition that two types of similar situations are possible--depending on whether the components for which the response operations are generated are similar (situations of unification (generalizing)) or distinct (situations of disjunction). This can be shown schematically as follows:

In situations of unification $A(a, b, c, \ldots, k, m)$---p;
$B(a, b, c, \ldots, k, n)$---p.
In situations of disjunction $M(e, g, h, \ldots, l, f)$---q;
$N(e, g, h, \ldots, l, s)$---r.

Here a, b, c, ...stand for the components of the material being studied; p, q, r stand for the response operations. In the future we shall call the learning material for which definite response operations are generated a complex stimulus. Similar situations are engendered by the content of curricular material or by the selection of practice exercises.

Example. In the study of inequalities it is established that the terms of inequalities can be transferred from one side to the other if the sign preceding this term is changed to its opposite. But it was the same way in the study of equations. Consequently, here we have similar situations of unification. On the other hand, when we study the multiplication or division of both sides of inequalities or equations by the same negative number,[12] similar situations of disjunction are generated. Actually, in solving the inequality $-2x > 3$ and the equation $-2x = 3$ (and in all analogous situations), the complex stimuli differ only in one component (the > and = signs) and are identical in all other components (the presence in the given expressions of two sides standing before and after the corresponding sign, the identity of these parts, and the identity of part of the problem's condition--to find the value of x). But in the solution, in the former case the inequality sign is changed to the opposite ($x < -\frac{3}{2}$), while in the latter, the equality sign is retained ($x = -\frac{3}{2}$).

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[12] In the textbook a general rule is given for multiplying both sides of an equation by a negative and a positive number. We are singling out the case of multiplication by a negative number in order to contrast it with the same case of multiplication of inequalities.

In the conditions of instruction similar situations sometimes follow one another directly and sometimes are separated by a time interval--occasionally a very considerable one.

Similar situations have not been the subject of special study in the mathematics education methodology. Even the concept of "similar situations" is lacking. In connection with this, the features of the pupils' mastery of mathematics remain unclear and there are no clear-cut recommendations on the teacher's working methods in these situations. Undoubtedly, teachers and pupils are sometimes more or less distinctly aware of the presence of learning material that is similar in certain features. It is also well known that in similar conditions pupils often confuse one thing with another; hence, very general recommendations often result--for example: "We must strive for complete understanding, clarity, and the like on the pupils' part." But the question of how best to do this remains open. For a working methodology in similar situations (which are very frequently encountered in mathematics teaching) to have sufficient basis, we must have a distinctive microanalysis of the instruction process, directed towards revealing the mechanism of mastering mathematics in these situations.

In this communication an attempt is made at such a micronalysis in similar situations of disjunction, on the basis of the author's experience working as a mathematics teacher, and on the basis of especially organized observation and experiments. Here we shall limit ourselves to a qualitative analysis of instruction in similar situations of disjunction.

The basic educational task facing a teacher in these situations comes down to forming in the pupils two different bonds of psychic processes (associations), each of which corresponds to one of the similar associations (for example, for multiplying both sides of an equation and of an inequality by a single negative number).

Let us agree to call similar the associations to be formed in similar situations of disjunction. All of the distinguishing components in the links in these associations (first or second) are identical, except for one. The formation of the pupils' ability to differentiate what is common and what is distinct in the learning material is evidently the chief feature in their mathematical preparation and is directly related to the formation of similar associations.

Pupils often make mistakes in similar situations. For example, in proving the similarity of certain triangles, they refer to the corresponding test for congruence instead of to the test for similarity: the difference of the cubes of two numbers is called the cube of the difference; the ratio of a leg to the hypotenuse is called the cosine instead of the sine; and the like. It can be observed that the essence of the matter here consists in the following. For many pupils, instead of the two necessary similar (and distinct) associations, only one arises, often functioning for a prolonged period--an association whose first link is an awareness of common features characteristic of the first two links of the necessary similar associations, but whose second link is the second link of one of these associations (usually the one that is simpler or the one that arose prior to the other).[13] Such an association is clearly erroneous, and its actualization can lead to errors in the pupils' operations. Let us agree to call these mistakes merging errors.

Example. The construction of graphs of the functions $y = ax^2$ for $a > 0$ and $a < 0$ is done in similar situations. To have a proper notion of the location of the graphs of these functions, the pupils should have formed two distinct (similar) associations, namely an awareness of the given expression $y = ax^2$ together with the sign standing before the coefficient of x^2 ($a > 0$ or $a < 0$) and an awareness that the branches of the parabola will be directed upward (or downward). In the first links of these associations, all components coincide (the presence of the argument x to the right after the equality sign, the coincidence of the values of the exponents of the argument, etc.), except for one--the sign before the coefficient of x^2. Objectively, a correct answer for the second case is possible only if there is a valent awareness of the minus sign. But it is well known that many pupils, particularly when first being taught to construct graphs of the functions $y = ax^2$ ($a < 0$), represent the corresponding parabola with the branches directed upward, that is, the same as for the case when $a > 0$.

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[13] Associations characterized by the indicated typical features have been noted in works by Shevarev [19], Yaroshchuk [21], and Artemov [1].

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But this means, first, that they are not valently aware of the minus sign, and, second, that in both cases the parabolas are constructed on the basis of the actualization of a single association, the first link of which can include only an awareness of the common features (or of some of them) of the expressions $y = ax^2$ for $a > 0$ and $a < 0$. The second term of this association constitutes the second term of one of the similar associations formed earlier (the pupils encounter the parabola of the type $y = x^2$ before their systematic study of the unit "Functions and Graphs"). The actualization of this association engenders a merging error in the construction of parabolas for $y = ax^2$ $(a < 0)$.

Many other similar examples from the practice of mathematics instruction could be cited. All of this provided a basis for formulating the following feature of mastering mathematics in similar situations:

> If 1) a definite response operation is generated for some complex stimulus; 2) a new response operation is generated for a new complex stimulus; 3) such complex stimuli have only one component apiece with which these stimuli are differentiated from one another and all other components identical, then 1) initially, the awareness of the distinctive component of the second complex can be invalent or seldom valent; 2) the awareness of identical components or of some of them can be fully valent (that is, such that a response operation depends only on these components).

Schematically: If the bond M(a, b, c, d)--m is generated and the bond N(a, b, c, 1)--n is formed, then a different bond N(a, b, c, 1)--m often arises, which engenders an error.

To diminish the negative influence of this peculiarity of mastering mathematics, it is essential to foresee which component of the newly introduced complex stimulus might turn out to be invalent the first time it is taught. For this purpose it is clearly essential to conduct a microanalysis of similar situations. The subsequent task consists in equalizing the valence of awareness of the distinct components of complex stimuli. We can suppose that the method of comparison is the most suitable for this. The use of it forces the pupils to differentiate the similar and the distinct in similar situations and will contribute to the formation of essential similar situations

The method of comparison has not found proper reflection in the mathematics educational and methods literature, and, in particular, it is not used in similar situations. In Larichev's problem book [11] there are a small number of exercises requiring that what is similar and what is distinct be established. However, a detailed analysis shows that they are inferior when it comes to the formation of similar associations. For example, in Problem 1777 one is asked to construct the graph of the functions $y = 2x + 3$ and $y = -2x + 3$, and to establish the similarity and difference in the constructed graphs. These exercises, as it is easy to show, are executed in similar situations. Consequently, to realize them, two similar associations should be formed or consolidated.

In the research of psychologists it has been established that for a needed association to arise and be reinforced in a pupil as a result of doing exercises, it is essential that the ongoing processes in the execution of these exercises strictly correspond to the first and second term of this association [19].

In the execution of the exercise that we are analyzing, the bonds of the processes constituting the terms of similar associations are not envisaged. The difference in the location of the straight lines is not placed in relationship to the sign preceding the coefficient 2. For essential associations to be formed, however, it is important to expose not only what is similar and what is distinct in the graphs but also to establish what engenders such distinction. Only in this case will the processes occurring in the execution of an exercise strictly correspond to the processes forming the terms of similar associations, and one can count on the formation of the necessary associations. This is just what is lacking in the exercise we have been considering.

Report VI: The Experimental Substantiation of the Methodology of Teaching Mathematics in Similar Situations*

In our preceding Report we noted that in the study of mathematics one often has to produce in the pupils distinct response operations for complex stimuli, similar in all components except one, namely: A(a, b, c, ..., k, m)---p and B(a, b, c, ..., k, n)---r. The situations in which response operations are produced for such complex stimuli were called similar. On the basis of a qualitative analysis of the results of observations it was established, for the instruction of pupils in similar situations, that if an association A(a, b, c, ..., k, m)---p is formed, and later (often after a protracted interval) an association B(a, b, c, ..., k, n)---r is formed, then initially the component *n* in the second complex can be seldom valent or invalent. On the other hand, the common components of both complex stimuli (or some of them) can be fully valent. The result is that instead of two essential (similar) associations, the pupils form one, the actualization of which can engender errors in operation (merging errors). It was suggested that in similar situations one should use the method of juxtaposition, in order to heighten the effectiveness of the instruction. In the present report, an experimental substantiation of this suggestion is set forth. All of the experiments were done by the method of cross-comparison [21].

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*Translated by Joan W. Teller.

First series (trial experiments)

The study in grade 9 of the graphic solution of systems of two linear inequalities and of linear equations presents similar situations.[14] In fact, the graphs of straight lines are constructed in both the former and the latter cases after appropriate transformations. But in solving a system of equalities the common values of the argument X for which both inequalities are satisfied are found *on the axis of abscissas*. In solving systems of equations, the abscissa and the ordinate *of the point of intersection* of the straight lines are found.

The topics named were studied in parallel fashion during two lessons.[15] The appropriate exercises were selected haphazardly--one or two on systems of inequalities, then on systems of equations, then on inequalities again, and so on. At the same time the pupils were required to clarify what was similar and what was distinct in the solution of inequalities and equations, and were obliged to disclose the reasons that gave rise to distinction in the operation. In other words, the instruction was done by the method of alternating juxtaposition and was intended to create in the pupils an objective of differentiating the similar and the distinct in similar situation In the third lesson written work was done independently, whereby one had to solve systems of inequalities and of equations graphically. Of 33 pupils, 3 made a merging error (i. e., approximately 9%) In finding the solution of a system of inequalities, they performed operations that were adequate for solving a system of equations.

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[14] The graphic solution of systems of inequalities is not directly specified in the curriculum. However, since it is not elaborate in content, it was done for experimental purposes.

[15] School Number 4 in Penza.

After this we spent two lessons, with the same pupils, reviewing the construction of graphs of quadratic trinomials and the disclosure, on graphs, of the increase and decrease of the given functions. In doing these exercises one had to compare distinct values of functions by comparing appropriate segments parallel to the y-axis. The situations arising here, of comparison of positive and negative values of functions, are similar. But, in contrast to the preceding case, there was no special comparison of what was similar and what was distinct in doing the exercises.

In the written work, which was also done in the third lesson, 13 of 33 pupils (approximately 39%) made a merging error. The error consisted in comparing segments without their direction into account. The greatest value of a function was always correlated with the segment of greatest length. This means that for pupils who made merging errors, instead of two similar distinct associations, only one erroneous association was formed and functione The awareness of the direction of the segments was invalent.

Later on (parallel with the study of the curricular material) a comparison of the values of functions based on graphs was repeated by means of alternating juxtaposition of the similar and the distinct and by disclosing the reasons giving rise to a distinction in the conclusions. In the repeated written work, which was done two weeks after the first, exercises were included on the graphic solution of systems of inequalities and equations and the comparison of the values of functions by graphs. Three out of 34 pupils made a merging error having to do with systems of inequalities and equations

(9%). It is apparent that the method of alternating juxtaposition promotes a sufficiency stable formation of necessary similar associations.

Seven persons (21%) made merging errors in performing the assignment on the comparison of the values of functions using graphs. If we compare this result with the result of the preceding work, we can observe that application of the method of alternating juxtaposition has resulted in an improvement in the pupils' abilities.

Second series

To exclude the influence of random factors (individual traits of the teacher and pupils, content of the learning material, and the like), a second series of experiments was conducted. The subjects were pupils in the eighth grade.[16] The experimental material was the content of the second and third tests for similarity of triangles. The study of these was done in situations similar to the study of the corresponding tests for congruence of triangles.

The experiment was done by the following method. In each school two classes taught by the same teacher were selected. In each class one test for similarity was studied using the method of contrasting it with a corresponding test for congruence (the method of juxtaposition); a second test for similarity was studied in the usual way -- without contrasting -- following the content of the textbook and the problem-book in geometry. For the study of both tests for similarity, two lessons apiece were set aside, without relying on the methodology of exposition. In the first lesson the content of the test and its proof were examined, and problems were solved; in the second, the problem solving was continued, and at the end of the lesson writte testing was done for approximately twenty minutes.

The juxtaposition was done according to this plan: a) Three pairs of triangles (congruent, similar, noncongruent, and dissimilar) were constructed to correspond to the test for similarity being studied. The pupils did this assignment at home. Then what was similar and what was distinct in the data for construction were clarified, as were the distinctions in the conclusions that engendered the distinctions in the data. b) After proof of the test for similarity, its content was juxtaposed with the appropriate test for congruence. c) Included among the exercises, for consolidation of the test for similarity was the solution of problems requiring the application of a corresponding test for congruence of triangles. For example, in the study of the third test for similarity by the method of juxtaposition, 7 problems were solved (including a homework assignment) before doing the test-work; 3 of these were on the application of the third test for congruence. In a parallel class--at the same time--all 7 problems covered only the third test for similarity in their content.

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[16] Schools Number 4 and Number 54 in Penza; teachers V. D. Sal'nikova and T. A. Polubarkina.

Among the tasks on the test were these:

1. The ratios between the three pairs of corresponding sides of two triangles are known. What can these triangles be?
2. In two triangles there is one equal angle, and the lengths of the sides containing these angles are known. What can the given triangles be?

Correct answers are possible only when distinct similar associations are present and actualized. An indication of three possible types of triangles (congruent, similar, unequal, and dissimilar) was regarded as a complete answer and signalled the presence of the necessary similar associations in the pupils.

The results of the experiments, formed by methods of statistics, are consolidated in Tables 2 and 3.

TABLE 2

EXPOSITION OF LEARNING MATERIAL USING METHOD OF JUXTAPOSITION

Number of Types of Triangle	Frequency	Total Observation	Mean	Standard Derivitive	Coefficient of Variation (percent)
3	57	171			
2	27	54	$\frac{249}{109}$		
1	24	24	≈ 2.28	0.84	36.8
0	1	0			
Total	109	249			

TABLE 3

EXPOSITION OF LEARNING MATERIAL WITHOUT JUXTAPOSITION

Number of Types of Triangles	Frequency	Total Observation	Mean	Standard Deviation	Coefficient of Variation (percent)
3	34	102	$\frac{195}{114} \approx 1.62$	0.99	61
2	23	46			
1	47	47			
0	10	0			
Total	114	195			

The less standard deviation and coefficient of variation, the better the results (on the average). A comparison of the means, standard deviation, and coefficients of variation speaks in favor of exposition using the method of juxtaposition.

When we calculate the reliability criteria for the difference t of the results of experiments in which the relationship of phenomena is studied (for us it is the relationship between the ability to differentiate the similar and the distinct, and the method of instruction), we obtain $t > 3$. It follows from this that the reliability of the difference according to the two teaching methods can be regarded as proved [20].

Conclusions

The validity of the previously formulated peculiarity of the mastery of mathematics in similar situations was confirmed. Actually, we proceeded from the fact that in similar situations some components of complex stimuli can prove to be invalent. During one case of instruction, measures were taken to increase the valence of awareness of such a component; in another this was not done. The results of instruction of the same examinees in the first case proved to be essentially better than in the second. Consequently, the content of this peculiarity reflects with objective correctness the course of instruction in similar situations.

Without the teacher's deliberate intervention, many pupils often form erroneous associations in similar situations; the formation of essential similar associations is done with considerable expenditure of time and effort. The method of juxtaposition is quite an effective means of forming similar associations. It can be realized in different variants. But in all cases the methodological task consists in making valent the awareness of a seldom valent or invalent component of a complex stimulus.

Whenever possible, a parallel study of learning material should be envisaged in similar situations with interchanged exercises on the formation of similar associations. If similar associations are separated by a significant time interval, it is advisable for the exposition of the learning material in the second situation to be done in juxtaposition with the learning material of the first, which undoubtedly is one effective device for reviewing material that has been covered. In this case the inclusion in the second similar situation of a small number of exercises (done at random) from the first (previously encountered) one promotes the formation of sufficiently stable and easily actualized similar associations. No additional expenditure of time is required for this (as experiments have shown).

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ON THE PROCESS OF SEARCHING FOR AN UNKNOWN WHILE SOLVING A MENTAL PROBLEM*

A. V. Brushlinskii

*Report I: The Role of the Problem in the Thought Process***

In the psychology of thought a generally correct thesis has long been firmly established--even in the Wurzburg school--to the effect that the course of solving a problem is determined primarily *by the problem itself* in its original formulation, in particular by the *requirements* of the problem [2, 7, 17]. In contrast to the problematic situation, a problem (the posing of an initial question, of a requirement, and so forth) immediately seems to establish an *initial determination* for thought, which determines the general direction of a search for the unknown. This thesis is largely true, but when it is given a one-sided treatment, it can be inadequate for a comprehension of the subsequent determination of thought. Very often this results in a metaphysical break between the initial and the subsequent determinations of the mental process. Such a break appears especially sharply if the problem is regarded only as a starting stimulus, just launching the thought according to the principle of an external incitement and then *in no way participating* in its *determination*. This interpretation of a problem, which originates, as we know, with Selz [15, 16], is retained--more or less obviously--in every psychological theory that is *limited* to a general thesis according to which the course of a problem's solution is determined by the problem itself, the direction of a search for an unknown is provided by the initial question, the requirement, and so on. (The same interpretation of the role of the problem in thinking has recently been revived in gnosiology, cybernetics, the so-called heuristics, and in other fields.)

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**Translated by Harvey Edelberg.

Such a break between the initial and the subsequent determinations of thought means, essentially, the reduction of the latter to the former. As a result the first stages of the thought process acquire a self-contained, exclusive significance, determining the course of thought, without contact with its subsequent stages. In the end, such an isolation leads to the elimination of any determined quality in thinking. A mechanistic approach to determination, as to the external incitement, in fact quite annihilates it.

A way out consists in understanding thought as a process, as an activity (of the individual). To treat it as a process, as Rubinstein [12, 13, 14] has done, means primarily to understand the determination of thought as a process (interactions between the initial, external and the specific, internal conditions of mental activity). This is a process of continuous interaction between the thinking subject and the knowable object, the objective content of the problem being solved. The determination of thought (as of any human action, in general) and its performance take place at the same time. It is not given indigenously as something entirely readymade; it is formed, it develops gradually--that is, it appears in the form of a process. Only in the course of thought itself are the specific, internal conditions for its further development created; the products or results of the thought are themselves included in it as preconditions for its subsequent course, and they become the means of subsequent analysis. The determination of the mental process is by no means formed just at some one stage (such as the initial one) or at a few stages of the thought process--in particular, in the case of an "insight" that arose no one knows how (guessing, etc.). All determination of thought is formed as a process--that is, it is formed continuously (but not necessarily uniformly) at all stages of mental activity.

To confine ourselves to one example: In the opinion of many psychologists, the solution of a problem involving the use of a general principle breaks down into two stages. In the beginning, in the first, creative stage, a principle is found that determines the subsequent course of solution; then in the second stage the principle is applied to the solution. This, in particular, is the position taken by Duncker [4], as Humphrey correctly observes [6].

This Gestalt psychology treatment of the relationship between the stages of thinking gathers up the determination of it into just one of its stages (insight). This difficulty is overcome if thinking is regarded as a process.

In Slavskaya's research [18] it is shown that the analysis of a problem and the actualization of a theorem (a principle for the solution) are not separate from each other, like two alien stages; they are intertwined, so that the general proposition (principle, theorem) and the particular conditions of the problem are continuously correlated with one another at every link in the mental process [14, 18].

Thus, it is utterly insufficient to limit oneself to a general thesis according to which the course of thought, the solution to a problem, is determined primarily by the problem itself (in its original formulation). The determination of thought is a process--that is, in the course of cognitive activity all new properties of an object are continuously revealed, as a result of which all new determinants, which determine the course of the mental process, arise. From this standpoint, a gradual isolation of what is being sought (the unknown) is essential, proceeding from its relations with the unknown in the problem, which (the relations) are manifested and analyzed step by step in the course of the thought. An unknown is not an absolute vacuum with which it is impossible to operate in any way. It exists in a definite system of relationships that connect it with what is already given (known) in the problem. For example, the requirement of a problem is to prove that three segments intersect at one point (under certain conditions, which we omit here, for simplicity). Special experiments show (see [3] for more detail) how, in the process of thinking, it becomes clear that three segments can intersect at three different points, and therefore it must be substantiated that these points coincide in one point.

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Proceeding from the dismembered relationships[1] (coincidence) between these points, as well as from other connections between the elements of the problem, one succeeds in isolating what is to be found--while still in a very approximate definition of it. The unknown here is certain "dimensions" of the parts into which the three segments are divided by their point of intersection (then it is clarified that the three segments are diagonals of parallelograms and consequently are divided in half when they intersect; this then leads to the solution of the problem). Therefore, we must distinguish between the requirement of a problem and the unknown. The former is given in the original formulation of the problem; the latter must be singled out gradually, in the process of solving it.

Report II: Distinguishing Between the Requirement of a Problem and What Is Being Sought*

In the psychology of thought and in pedagogy [1, 10, 11] one usually identifies a problem's requirement and what is being sought, the finding of which constitutes a solution to the problem. Special analysis of this question, however, leads to the conclusion that such an identification is invalid.

In the requirement of a problem (insofar as it is distinguished from a problem situation--see [5, 8]) definite points of departure for isolating and describing the unknown are indicated or sometimes given directly. But the unknown (what is being sought) is such that it is not given, but only specified by the initial conditions and the requirement of the problem. Only a part of what is being sought is outlined, as it were, in a problem's requirement (question): The entire unknown is specified by the entire problem. Therefore it is impossible to identify what is being sought with the requirement by reducing the former to the latter. This is especially clear with respect to problems on proof.

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[1] We know from the history of psychology that, in itself, the role of relationships (between elements of a problem) for the solution of a problem has been noted repeatedly. For example, Selz [15, 16] compares a problematic complex with an uncompleted form or diagram, in which there is a blank that needs to be filled in. This blank can be filled in by having due regard for the relationships of this unknown to the known components of the complex. However, an analysis of Selz's interesting work shows that these relationships usually turn out to be in the previous data, known from the formulation of the task. In reality, the person who solves a problem should himself discover these relationships, which are the starting point for subsequently determining what is to be found. In the experiment we paid particular attention to this discovery of relationships by the examinees.

*Translated by Joan W. Teller.

In the requirement of a problem on proof, a proposition that is subject to substantiation is formulated directly. What is being sought, consequently, is not this given (known) proposition, but only its logical basis, about which nothing is stated directly in the problem. Here the problem's requirement and what is being sought are separated in an obvious way.

The distinction between what is being sought and the requirement (the question) can appear in a way that is not so obvious in problems on computation (in which one is required to compute, for example, the weight of an indicated object). But even then what is being sought is not, of course, the technique of calculation, not the computational operations of addition, multiplication, etc. (themselves not requiring thinking), but the general relationships (physical, mathematical, etc.) that are contained only implicitly in the given concrete problem and on the basis of which the entire technique of calculation is then performed almost automatically. What is being sought in the problem is *not the definite*, specific weight, volume, etc., of a specific object, but rather the general essential interrelations between the weight and the other properties of objects appearing in the particular case, which are not revealed by the original formulation of the problem.

In the formulation of any problem (in contrast to a problem situation), the initial conditions and the requirement are *given*. If they are given, then they do not have to be sought. We need to search for their foundations, causes, consequences, interrelationships, etc., about which nothing is stated in the initial formulation of the problem. These constitute what is being sought.[2]

The basic point of our distinction between a problem's requirement and what is being sought consists in the following: A problem's requirement can be given even, as it were, from without, from the side (by an experimenter, an instructor, a practical worker who confronts the scientist with a problem, and so on); by contrast, what is being sought is distinguished and formulated only by the person solving the problem (of course, the latter, in a number of cases, can also formulate a problem's requirement himself).

A view that what is being sought and a problem's requirement are identical is a natural consequence of a restricted understanding of the latter. According to such a restricted understanding, the process of solving a problem is determined by the problem itself (in its initial formulation). Here the initial, opening determination of thought is made absolute, with an underestimation of the determination following it, which develops in proportion as the problem is transformed. In its extreme expression what is being sought and the problem (in its original formulation) are identical. As a result, subsequent determination of thought may turn out to be less essential or altogether superfluous, since the initial formulation of the problem--entirely by itself--predetermines the entire subsequent course of the cognitive activity. The determination of thought as a process is not envisaged in the least here. Such a predetermining, proceeding from the initial formulation of the problem, can result in a complete coincidence of the characteristics of what is being sought, the unknown, with the way in which it is directly characterized in this formulation (the explicit content of the formulation). In other words, more and more meaningful definitions of what is being sought, which are obtained only gradually and with difficulty in the course of the *entire* mental process of solving a problem, with such a notion of its structure, are directly identified with characteristics of an unknown that are immediately given and still quite indefinite, being contained only in the *initial* formulation of the problem, i.e., in particular, in the problem's requirement. The conception of a problem's *requirement* and what is being sought as identical is therefore a particular case of the conception of the *problem* and what is being sought (replacing what is being sought by a problem) as identical.

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[2] We need to make a distinction not only between the requirement of a problem and what is being sought, but also between what is being sought and the unknown. Not every unknown becomes what is being sought straight away (but, of course, everything that is being sought is at first unknown). This appears especially distinctly when arithmetic problems are solved algebraically, when one designates the "unknown" by X and operates with it as something known, during the search for other unknowns, which have become what is immediately being sought. In the requirement of problems on computation such an unknown can be indicated, but it is not what is being sought in the proper sense of the word. In some cases (see below) such a distinction between what is being sought and the unknown can be disregarded.

Viewing a problem's requirement and what is being sought (the unknown) as identical leads to an inadequate understanding of the latter. Many authors, adhering to the idea of such a view [9, 10], suppose that in the thought process one must be guided by the following heuristic rules: "Look at the unknown," "We must concentrate on the unknown," "From the very beginning one must clearly see what is being sought," and so on. This is just the same as advising a blind man to look carefully ahead. Though the unknown, what is being sought, is really unknown, we are still faced with somehow selecting, gradually isolating all of the richness of its attributes, etc. It is simply impossible to see at once and *clearly* "what is being sought." If it were possible, then there would be nothing to look for or to solve; no problem would remain.

Other so-called heuristic rules are just as inadequate: "Look at what is known" and "Look at the unknown," "Transform the unknown elements" and "Transform the given elements" [10]. Consequently, it is suggested that both the known and the unknown be subjected to *identical* operations ("examining," "transforming," etc.). As it turns out, one can operate with the unknown *in exactly the same way* as with what is known. Here every *specific feature of the unknown* that distinguishes it from what is given disappears. Thus the matter of the question (which as it were, introduces or projects the unknown is ignored, and the matter is primarily psychological. It is no coincidence that logic could not even approach it.

What is being sought and the requirements of a problem are made identical first by replacing the former by the latter. For example, the authors of the above-mentioned rules of thought [10], without noticing themselves, usually mean, in reality, by an unknown or what is being sought precisely the requirement (question) of a problem. But the requirement of a problem is always known, since it is given in its initial formulation. Then the only unknown here turns out to be only the connection between what is being sought and what is given, since what is being sought here is replaced by the requirement of a problem, that is, what is given, known. The source and the erroneousness of the widespread view by which everything is reduced merely to the necessity of revealing connections between what is being sought and what is given are thereby revealed. Such a reduction can assume that what is being sought is already determined, and one has only to find its connections with what is known or given. That is, first what is being sought is found--outside this connection, as it were, and then the connection as well is found. In reality, the one is determined only by the other in the process of analysis through synthesis.

Thus three positions that might at first glance seem quite remote from one another open to criticism come together here: a) the course of solving a problem is determined by the problem itself; b) what is being sought is identical with the requirement of a problem; c) in the course of solving a problem one must discover (only) the connection between what is being sought and what is given. These views can be surmounted if we regard the determination of thought as a process--in the sense of the word as we designated it in our initial report.

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THE MECHANISMS OF SOLVING ARITHMETIC PROBLEMS

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Much psychological and methodological research (Menchinskaya [5, 6], Kalmykova [4], Yaroshchuk [12], Talyzina [10], Schedrovitskii [9], et al.) has been done on the questions involved in solving arithmetic problems. This work includes material that describes the solving of arithmetic problems by experimental subjects of various ages---from pre-schoolers to adults. However, no integral theory of the mechanisms of this process has yet been created. There is reason to assume that one of the essential obstacles in the creation of such a theory is the very method of approaching the establishment of mechanisms of solving the problems.

In the first place, the problem-solving processes are most often investigated with subjects who have somehow already learned how to solve problems (and not necessarily in school). As a matter of fact, a fully determined mechanism of problem solving is investigated, which is the consequence of certain methods of instruction, established historically in the methodology of arithmetic. But is this mechanism optional? It is possible to posit other methods of learning, which yield other mechanisms of solving problems and different results from those which have been investigated?

Second, many investigators do not define precisely the concept of a problem in general and of an arithmetic problem in particular, assuming, evidently, that a "problem" is something simple and generally known and therefore does not need a special definition. At best the following definition, widely used in methodology, is cited: "An arithmetic problem is a question in which one is to use arithmetical operations to find an unknown number (or numbers) according to given numbers [1:67]." Such a definition can mean almost any problem, not just an arithmetical one. By relying on it, it is difficult to construct a psychological investigation of mental activity.

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Considerable difficulties are also related to the questions of the essence of the solution of arithmetic problems. This process can be considered from various points of view: mathematically--which mathematical operations should be performed in order to answer the question of the problem; logically--of which logical operations does the process of solving problems of various types consist; psychologically--of which mental operations does the solution process consist; educationally--what are the teaching devices for developing an ability to solve the problems in the pupils? Investigating the solving process from the psychological or educational aspect, one must present clearly the essence of this process from the standpoint of logic and mathematical features of arithmetic problems only vaguely. A special and careful examination of this aspect of the problem is necessary as an important prerequisite to psychological research into mechanisms of solving.

We have conducted an analysis of the logical-mathematical characteristics of arithmetic problems and have constructed a hypothesis about the mechanisms of solving them [3]. Underlying it was a distinctive methodology of teaching the solution of arithmetic and algebra problems, which has been tested experimentally for several years in a number of schools in the Sverdlovsk province, under the direction of Semenov [7, 8] and in the Tadzhik Soviet Socialist Republic, under the direction of Asimov [2] and Turetskii [11]. The positive results of these trials permit us to think that the hypothesis we have advanced describes correctly the basic characteristics of the mechanisms of solving arithmetic problems. In a short article there is no opportunity to explain in detail the whole course of the analysis of the problem and of its experimental elaboration (this has been done in enough detail in a number of works [3, 8, 11]). We shall dwell on several questions of principle, expecially significant, in our opinion, for future, strictly psychological investigations in this area.

The generic concept of an "arithmetic problem" is a "problem situation," by which we mean the following. The quantitative aspect of any phenomenon (process, event) of reality is characterized by many quantities, each of which assumes a certain value at a given moment. To characterize the quantitative aspect of a phenomenon there is no need to know the values of all the quantities, for they are bound together by several relationships. Based on known values of some quantities we can find out (calculate) the values of other quantities, which characterize the same phenomenon. We call this situation the "problem situation." To calculate the unknown values of some quantities that characterize such a situation, one must first translate it into mathematical language or, as they say, construct a mathematical model of it. Constructing such a model of a real situation on a psychological level is not the act of a single moment, but a complex process of many steps, which involves constructing a whole sequence of models of the situation from the simplest ones to the most abstract, rigorously mathematical (symbolic) ones. In this sequence three basic types of models can be isolated:

I. The object-model: It is made of any standard objects (small sticks, cubes, nuts, etc.) and represents the values of the quantities and their interrelationships on the basis of an actual reproduction of the operations that characterize these relationships.

We regard the object-model as dynamic. Of course, it can also be treated as the construction of a chain of static models--here the transfer from one link to the next is made in conformity with the operations that characterize these relationships.

II. The verbal model, or strictly arithmetical problem, of which the verbal specification of individual values of quantities and the verbal specification of relationships between these values are the basic, primary elements.

The verbal specification of individual values usually consists of the following three parts: 1) the name of the quantity to which a given value refers; sometimes the name itself is omitted, but then the units of measure in which the value is specified must be indicated, so that it will be easy to establish the name of the quantity, when needed; 2) the indication of characteristics of the given value that distinguish it from other values of the same quantity; 3) the numerical extent of a given value, if it is known.

We shall illustrate this with the following problem:

> The distance from A to B on a railroad is equal to 200 kilometers, and from B to C it is 400 kilometers. What is the distance on the railroad from A to C, passing through B?

In this problem one quantity is considered--the path (distance) specified by three of its values. The first value is specified as: "The distance from A to B on a railroad is equal to 200 kilometers." It consists of three parts: 1) "the distance"--the name of the quantity; 2) ". . .from A to B on a railroad"--the indication of the characteristics of the given value; 3) "200 kilometers"--the numerical extent of the value.

The second value of the quantity under consideration is specified: "and from B to C it is 400 kilometers." Here there are the same three parts, but the first is understood because it is the same as for the first value. The third value is specified somewhat differently: "What is the distance on the railroad from A to C, passing through B?" Here there are two of the three parts: 1) ". . .the distance"--the name of the quantity; 2) ". . .on the railroad from A to C, passing through B"--the indication of characteristics of this value. The third part--the numerical extent--is missing. This shows that the given value of the quantity is **unknown**. Since the question "What?" is included in its specification, this value is the unknown that we are searching for.

Thus, if the specification of the value of a quantity includes all three parts, then this value is known (given); if the third part (the indication of numerical extent) is lacking in the specification, then that value is unknown. Unknown values of quantities occur in three types:

a) the desired ones, the extent of which we are to find; finding the numerical extent of these unknowns is the immediate aim of solving arithmetic problems. That the given unknown value of the quantity is sought for is easy to establish by the very method of specifying this value: The specification always includes an indication of the need to find the numerical extent in the form of a question asking "How much?" or something equivalent.

b) auxiliary or intermediate unknowns, the extent of which does not have to be found, according to the question, but one can and should find them while solving the problem.

c) the undetermined unknowns, the extent of which is also not required and cannot be found, in the conditions of the problem, but without them it is impossible to establish ties--correlations with other values of the quantities.

Let us examine examples of these values of the quantities,using the text of the following problem:

> One worker working alone can finish a task in 6 hours. A second worker working alone can finish the same task in 4 hours more than the first. In how much time can both workers together finish this task?

Here three values are examined: the time spent on the task, the amount of work, and the productivity of the workers. The first quantity was assigned three values: a) a known value for the first worker's time (6 hours); b) an unknown auxiliary value for the second worker's time; and c) a desired unknown value for the time of their combined work. The indication ". . . in 4 hours more than the first" is the value of the quantity of the difference relationship between the first two values and serves to connect them.

The second quantity (the amount of work) is assigned one value without an indication of the numerical extent ("a task," "the same task," "this task"). Therefore it is unknown. In its verbal specification the question of "How much?" is not included, but an analysis of the problem's content shows that the extent of this value cannot be found from the text of the problem, and that, consequently, this is an undetermined value. The third quantity (productivity) is not even named, but since the problem specifies quantities for the amount of work and the time spent on it, the quantity of their relationship is thus also specified--the productivity of the workers. It has three values: the first worker's productivity, the second worker's, and the productivity of their combined efforts. All of these three unknown values are undetermined, on account of the undetermined value of the quantity of the amount of work.

A little further on, we shall examine the methods of specifying relationships between the values of quantities in arithmetic problems.

III. The mathematically symbolic model of real situations has two aspects: 1) a numerical formula or group of numerical formulas, according to which one can calculate the numerical extent of the desired values of quantities; 2) an equation or system of equations, the solution of which yields the numerical extent of the desired values of quantities. We shall call the first aspect of the mathematical model an arithmetical model of this situation, and the second--an algebraic model.

All other models that are constructed during a transition from a real situation to a mathematical expression of it are the essence of a different kind of modification of the three types of models indicated above.

Let us examine elementary forms of situations, the mathematical model of which can be written with the help of a numerical formula containing only one arithmetical operation. In human activity many such situations arise, but they can all be reduced to several basic groups. These groups have been selected over centuries of human practical activity. All elementary situations can be subdivided into three groups.

Group I: Situations that arise as a result of certain operations with objects. Depending on the nature of the operation,the following types are possible here:

Type 1 is characterized by the operation of combining several values of quantities (objects) into one value or of two sets of objects into one set;

Type 2 is characterized by the operation of taking away (subtraction), which is the reverse of the operation of combining;

Type 3 involves the operation of transfer from a large unit of calculation (or measurement) to a smaller unit;

Type 4 is related to the reverse operation of transfer from a smaller unit of calculation (or measurement) to a larger.

Group II: Situations that arise as a result of the comparison of two values of a single quantity. If the values turn out to be equal, then we have a situation of equality; if they are not equal, then we obtain one of two types of situations, depending on the method of comparing the values : 1) comparison by finding a difference relationship, and 2) comparison by finding a multiple relationship.

Group III: Situations that arise in the quantitative description of one feature of any phenomenon by means of several interrelated quantities, each of which assumes, at the moment under consideration, a certain value (known or unknown). This group is divided into types depending on the quantities that characterize a given phenomenon; chief among them are distance-time-speed; value-quantity-price; amount of work-time-productivity; etc.

The verbal models of elementary situations we call correlations, of which the terms are the values of the quantities specified by the verbal method. Correlation models of the situations in group I we shall call correlations of operations, models of the situations in group II--correlations of comparison, and, finally, models in group III--relationship correlations.

The three values of a single quantity are terms in each correlation of operations, of which at least one value is unknown. The term in the correlation which corresponds to the result of the operation we call the main term. A characteristic feature of the verbal method of specifying the correlations of operations is the presence of special word-signs or of a main term. Thus, in the correlation of operations of combining, there is the phrase, "in all" or a synonym of it; in the correlation of operations of transition from one unit of calculation (measurement) to another we find the phrase "in all" in conjunction with the prepositions "at" or "with," included in the specification of one of the non-central terms of the correlation. Let us note that the correlation of the operation of taking away is specified in the same way as the correlation of the operation of combining. There are no special differences between the specification of the correlation of the operation of transfer from a larger unit of calculation (measurement) to a smaller, and the opposite operation. Therefore in the first group only two types of correlation are actually different: 1) the correlations of the operation of combining or taking away, and 2) the correlations of the operation of transfer from one unit of calculation (measurement) to another.

The values of one quantity and the result of comparing them in the form of difference or multiple relationships (if they are not equal) are the terms of the correlation of comparison. Here the results of the comparison is the main term. The verbal specification of correlations of comparison is also characterized by the presence of special word-signs: "so much as," "so many more (or less) than."

The values of three different quantities, bound by a given type of relationship, are the terms of the relationship correlations. The main term is the term whose value is equal to the product of the other two terms. No special word-signs enter into the verbal specification of these correlations, and we distinguish them only by the names of the quantities that are bound by a certain relationship.

As was already noted, just one of the terms of any correlation is unknown. The correlation having only one unknown term we shall call solvable. If there are two or more unknown terms, then the correlation is unsolvable. If the unknown term of the correlation is what we are looking for, that is, if its specification includes the question "How much?" or an analogous question, directly indicating the need to find the numerical extent of this term, then we call such a correlation central. A correlation which is solvable and central forms a simple arithmetic problem. Unsolvable correlations can only be component parts of complex arithmetic problems.

Let us give examples of simple arithmetic problems of the different types:

1. Fifteen rubles was paid for a child's table, and 5 rubles for a chair. How much did the table and chair cost together? (A simple problem of the operation of combining or taking away; the unknown is the main term.)

2. Vitya had 18 apples in all; he kept 5 apples for himself, but he gave the others away to his sister. How many apples did Vitya give her? (A simple problem of the operation of combining or taking away; the known number of apples belonging to Vitya is the main term.)

3. Some Young Pioneers planted 3 rows of apple trees, with 6 trees in each row. How many apple trees did they plant in all? (A simple problem of the operation of transfer from one unit of calculation to another; the main term is unknown.)

4. Some pupils gathered 180 kg of potatoes. They put all the potatoes in three sacks, an equal number in each sack. How many kilograms of potatoes did they put in each sack? (A simple problem of the same type, but the known value of the weight of the total number of potatoes gathered is the main term.)[1]

5. A boy dug up 10 cucumbers from one bed, and from another he dug up half as many. How many cucumbers did he dig up from the second bed? (A simple problem of multiple comparison; the main term is the known value of the abbreviated relationship.)

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[1] It is also possible to interpret any problem involving the operation of transfer from one unit of calculation (measurement) to another as a simple relationship-problem. For example, the correlation specified in the last problem (4) can be interpreted as a correlation-relationship between the number of sacks, the weight of each sack, and the total weight of all the sacks.

6. Five kg of granulated sugar was bought at 90 kopeks per kg. How much did the whole purchase cost? (A simple problem-relationship, the main term is unknown.)

We have described the types of simple arithmetic problems. We shall examine the mechanisms of solving them. Let us note that by the "mechanism of solving a problem" we mean the normative algorithm of solving which is the consequence of the logical-mathematical analysis of problems, taking into consideration certain results of psychological investigations of the mechanism of solving these problems. To put it more simply, it is our understanding of how pupils should solve these problems. This, as a matter of fact, is the "tentative basis" of those operations (in Gal'perin's terminology) which the pupils should perform in order to solve the problem. Of course, the actually observable mechanisms of problem solving, that is, mechanisms in a psychological aspect, will not fully coincide with the "ideal" mechanisms as set forth below.

However, the "ideal" mechanisms of problem solving are also different depending on the extent of the pupils' mastery of the methods of solution, on the pupils' stage in the learning process. In the lengthy process of instruction in the solution of simple arithmetic problems, the following stages can be outlined.

Stage 1. In this stage it is not the arithmetic problem itself but the real situation or its object-model that is initial and primary. The situation is resolved "with objects," that is, by means of an actual execution of the operation and a counting of the objects. The child learns, by imitating the person who is teaching him, to make a verbal model of the situation (i.e., to compose an arithmetic problem based on a real situation or an object-model of it), and he gradually comes to recognize the meaning of the first two arithmetical operations as the mathematical model of the situations of the first two types of group I (these types will usually be the only ones examined here). Parallel with this, the child masters counting, and then the method of performing the first two arithmetical operations with the aid of direct and reverse counting. Thus, the child still is not solving arithmetic problems here--he is only being prepared for it.

Stage 2. In this stage the simple arithmetic problem in its usual form becomes the starting point. Solving is done in approximately this way: For a given problem an object-model of the situation is constructed, and then during the actual performance of the modelling operation in object form, the problems receive an answer. Then, by means of counting the objects, a numerical answer is found, with the problems usually limited to the first type in group I. At this stage the child learns chiefly how to construct object-models of simple arithmetic problems.

Stage 3. According to a specified simple arithmetic problem, the construction of an object-model occurs, accompanied by a transformation of the verbal problem into its normal form, which contains only the numerical data of known values, the unknown, and the word-signs for the given type of correlation. Thus, the normal form of problem 1, cited above, will be: "15 rubles and 5 rubles. How much altogether?" The transformation of the problem into its normal form is done by establishing the omitted parts of verbally specified terms of the correlation, the word-signs, by generalizing the subject matter of the problem, and by abstracting oneself from all the particular characteristics of individual values and their interconnections. Having obtained the normal form of the problem, one can then construct its arithmetical or algebraic model.

Stage 4. This differs from the preceding stage in that the construction of the object-model occurs in the imagination but not actually. However, the transformation of the problem into its normal form and the subsequent transfer to an arithmetical or algebraic model is done in detail.

Stage 5. The fifth stage is characterized by the curtailing of the process of transfer from the set problem to its arithmetical or algebraic model. In the final analysis this transfer, that is, the solution of a simple arithmetic problem, is the act of one moment.

Let us examine the question of the solution of a complex arithmetic problem. It can be defined as a system of interconnected correlations, of which at least one is central, satisfying several requirements [3]. The type of complex problem, the method of solving it is largely determined by the nature of the connections between the correlations included in this problem, that is, by the problem's structure. To study this structure, one must find such an apparatus as would reveal the connections between correlations, would make them maximally graphic. Structural models in the form of diagrams of problems are such an apparatus.

They are constructed in this way. We introduce designations for the terms of the correlation: Let us designate the known terms by rectangles in which are written the magnitude of the terms; let us designate the unknowns by circles; let us designate the auxiliary unknowns by triangles and,finally, the undetermined unknowns by rhombuses. We agree to designate each correlation by a closed figure (an extended rectangle or a curvilinear figure), in which the designations of the terms of the given correlation are placed, joined by the sign for the appropriate operation and by an arrow directed toward the main term. Figures 1A and 1B show structural models of problems 1 (operation of combining and 6 (a simple problem-relationship).

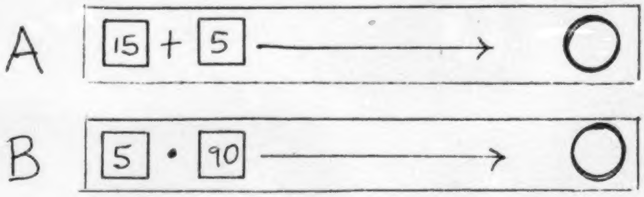

Figure 1

Let us examine the construction of the structural models of complex problems.

7. In a store Ivan bought 3 kg of cookies at 90 kopeks per kg, 4 kg of sugar at 1 ruble 5 kopeks per kg, and several kg of groats at 60 kopeks per kg. Knowing that Ivan paid 10 rubles 50 kopeks for the whole purchase, how many kg of groats did he buy?

Let us analyze the text of the problem: 1) the event examined in the problem is described by three quantities: cost-weight-price; 2) the cost is specified by four values: three auxiliary unknowns (the cost of the cookies, of the sugar, and of the groats) and one known (the total cost of the purchase); the weight is designated by three values, of which two are known (the weight of the cookies and the weight of the sugar) and one is unknown (the weight of the groats); the price is specified by three known values (the prices for the cookies, the sugar, and the groats); 3) all ten values of the three quantities are connected by four correlations: Three correlations are the relationships between the values of cost, weight, and price of the cookies, sugar, and groats, and one is the correlation of the operation of addition between all four values of cost. If we designate all of these ten values by symbols and combine them in the indicated correlations (with the unknown values entering into some of the correlations, i.e., they are signal stations between correlations), we obtain a diagram that is a structural model of this problem (Fig. 2A).

Figures 3A and 3B show the structural models of two more problems:

8. One brother is 5 years old, and the other is 4 times older than he. How old will each of the brothers be when the older brother becomes only three times as old as the younger?

9. One boy has 3 times as many nuts as another, and all together they have 48 nuts. How many nuts does each boy have?

The models of complex problems are diagrams consisting of "segments" bound between them--models of individual correlations. The models of solvable correlations are "segments" with one unknown "point," and

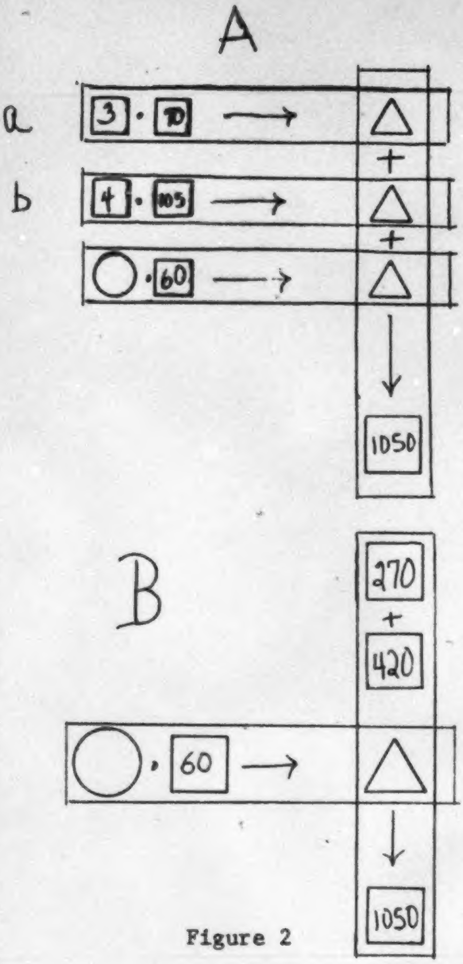

Figure 2

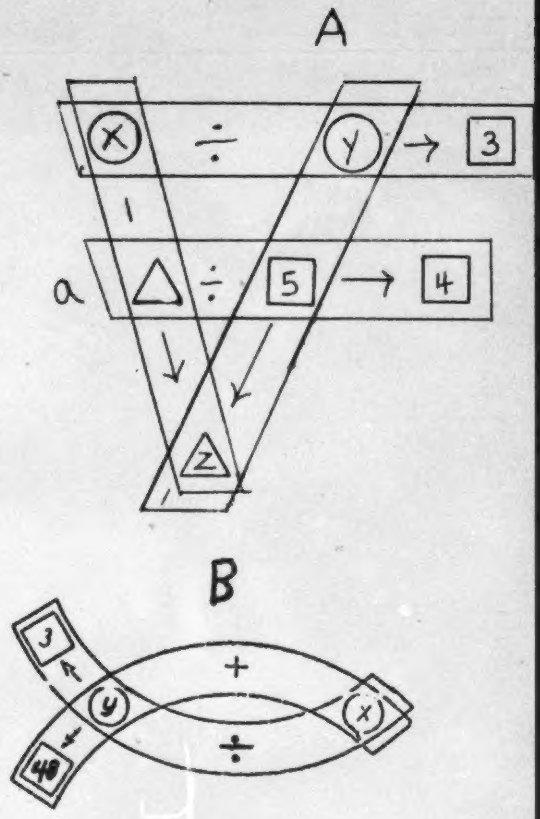

Figure 3

models of unsolvable ones are "segments" with two or more unknown "points." "Segments" with one unknown "point" we shall call entrances of a diagram. The diagrams of problems 7 and 8 have entrances, but the diagrams of problem 9 do not. Problems whose diagrams have entrances we shall call open, and problems whose diagrams do not have entrances we shall call closed. Each entrance segment of the diagram of a complex problem can be dismembered, and then the correlation corresponding to it forms a simple problem; after having solved it, we convert the unknown point of this entrance segment into a known point. But this point, because it is a signal station of the diagram, has entered into other segments as well. When it has become known, the segments that have not hitherto been entrance segments might now become such.

This process of dismembering the entrance segments can be prolonged. For instance, the diagram of problem 7 (Fig. 2A) has two entrance segments (designated in the figure by a and b). If we dismember them, we obtain these simple problems:

1) Ivan bought 3 kg of cookies at 90 kopeks per kg. How much did all the cookies cost?

and

2) Ivan bought 4 kg of sugar at 1 ruble 5 kopeks per kg. How much did all the sugar cost?

Solving them and replacing previously unknown values of the cost of the cookies and the sugar by the numbers we have found, then eliminating the solved correlations, we obtain this problem:

In a store Ivan bought cookies for 2 rubles 70 kopeks, sugar for 4 rubles 20 kopeks, and several kg of groats at 60 kopeks per kg. Ivan paid 10 rubles 50 kopeks for the whole purchase. How many kg of groats did he buy?

The diagram of this transformed problem (Fig. 2B) contains an entrance segment. We shall isolate it in the form of a simple problem:

In a store Ivan bought cookies for 2 rubles 70 kopeks, sugar for 4 rubles 20 kopeks, and some groats with the rest of his money. He paid 10 rubles 50 kopeks for the whole purchase. How much did the groats cost?

Solving this problem and eliminating the solved correlation from the transformed problem, we obtain this problem:

Ivan bought 3 rubles 60 kopeks' worth of groats at 60 kopeks per kg. How many kg of groats did he buy?

Having solved this simple problem, we shall find the unknown of the original problem.

If the same method is applied to problem 8, then after the calculation of the first entrance segment (designated in Figure 3A by the letter *a*), this process is interrupted and the closed problem will remain, but still simpler than the original one.

To solve the open problem, we should apply a purely *arithmetical* method of consecutive calculations of the simple problems, which in the majority of cases leads to a complete solution, and in the remaining cases simplifies it essentially. To solve closed problems, it is advisable to apply an *algebraic* method of solution, using this general rule: 1) designate each unknown point of the diagram by a special letter; 2) write out each segment of the diagram in the form of an equation; 3) solve the resulting system of equations, after first shortening it to a simpler system or to one equation. For instance, solving problems 8 and 9, we obtain these systems:

$$\left.\begin{aligned} x \div y &= 3 \\ x - 20 &= z \\ y - 5 &= z \end{aligned}\right\} \qquad \left.\begin{aligned} x \div y &= 3 \\ x + y &= 48 \end{aligned}\right\}$$

Of course, there is no need for the students to make such a detailed solution and especially to construct a structural model for literally all problems. Such a solution is needed only in the *first period* of instruction. As soon as the pupils learn the methods of solving complex problems, individual elements of this solution are curtailed and fall away. In particular, the construction of the diagram drops out; it is then possible to compose a curtailed system of equations or even one equation immediately. Composing diagrams of arithmetic problems is not only a good means for teaching how to solve them but is also an important condition for developing methods of a *theoretical* analysis of them in the pupils.

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